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Relation Algebras, Matrices, and Multi-Valued Decision Diagrams

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Presentation Outline

Concrete Relation Algebra

Definition

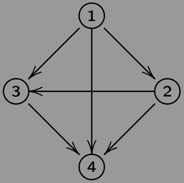
A concrete relation R between two sets A and B is a subset of the Cartesian product $A \times B$, where

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

[3] [1]

Example

Let $A = \{1, 2, 3, 4\}$ and let R be a relation defined on A such that $R = \{(a, b) \mid a < b\}$, then the relation can be visualized as set given by:
$R = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$.
R can also be visualized as a graph, a table or a boolean matrix as shown in the figures below:



R	1	2	3	4
1		×	×	×
2			×	×
3				×
4				

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

RelView System

"RELVIEW, developed in Kiel (formerly in Munich). RELVIEW is an interactive tool for computer-supported manipulation of relations represented as Boolean matrices or directed graphs, especially for prototyping relational specifications and programs " [2].

- RELVIEW implements relation in the form of boolean matrices using Ordered Binary Decision Diagrams (OBDDs).
- OBDDs are an efficient data structure used to implement very large Boolean functions.
- The first versions of RELVIEW were implemented using only arrays.
- A BDD is a directed acyclic graph (DAG) with decision nodes and two terminal nodes (0 and 1). Each decision node is labeled by a Boolean variable and has two child nodes (low child and high child).
- A BDD is called 'ordered' if different variables appear in the same order on all paths from the root to the leaf node. A BDD is said to be 'reduced' if there is no isomorphic subgraphs .

Heterogeneous relation algebras

A variant of the theory of relation which originated from category theory is called heterogeneous relation algebras. In this approach, relations have different source and target.

Definition

A (heterogeneous abstract) relation algebra is a locally small category $\mathcal{R}$ consisting of a class $Obj_{\mathcal{R}}$ of objects and a set $\mathcal{R}[A, B]$ of morphisms for all $A, B \in Obj_{\mathcal{R}}$. The morphisms are usually called relations. Composition is denoted by ";", identities are denoted by $\mathbb{I} \in \mathcal{R}[A, B]$, conversion
$\smile_{AB}: \mathcal{R}[A, B] \to \mathcal{R}[B, A]$
The operations satisfy the following rules:

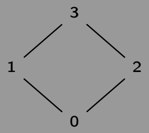
1. Every set $\mathcal{R}[A, B]$ carries the structure of a complete atomic boolean algebra with operations, $\sqcup_{AB}$, $\sqcap_{AB}$, $\overline{}_{AB}$, zero element $\perp\!\!\!\perp_{AB}$, universal element $\top\!\!\!\top_{AB}$, and inclusion ordering $\subseteq_{AB}$.
2. The Schröder equivalences

$$Q; R \subseteq_{AC} S \iff Q^{\smile}; \overline{S} \subseteq_{BC} \overline{R} \iff \overline{S}; R^{\smile} \subseteq_{AB} \overline{Q}$$

holds for all relations $Q : A \to B$, $R : B \to C$ and $S : A \to C$.

Example of Heterogeneous Relation Algebras

- Two Boolean algebra $\mathcal{L}_2$ and $\mathcal{L}_4$ with 2 and 4 elements.



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Identity	
A	1
B	1

Bottom	
A, A	0
B, B	0
B, A	0
A, B	0

Top	
A, A	3
B, B	3
B, A	1
A, B	1

- Similarly, we can produce the tables for meet, composition, converse and complement.

- Union Operation

A, B	0	1
0	0	1
1	1	1

B, A	0	1
0	0	1
1	1	1

A, A	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	0	3	2	3
3	3	3	3	3

B, B	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	0	3	2	3
3	3	3	3	3

Problems and Motivations

1. Splittings exist in matrix algebras
2. Standard and Non-Standard models of relation algebras
    - (Point axiom) For every relation $R \neq \perp\!\!\!\perp$ there exit two points x, y such that $xy^{\smile} \subseteq R$
    - Concrete relations between sets form the standard model of relation algebras. But the theory has non-standard models.
    - RELVIEW works with the standard model. This system cannot visualize computations in non-standard models of the abstract theory of heterogeneous relations
    - That is, not all relation algebras can be represented as the algebra of Boolean matrices.

Example

- The relation $\top\!\!\!\top ; \top\!\!\!\top = \top\!\!\!\top$ is always true in RELVIEW since it uses the standard model. These properties might not be true in some non-standard models, hence RELVIEW is not a complete system for relation algebra.
- An advanced example is given by the relationship between the power set of a disjoint union of two sets A and B and the product of the power set of A and the power set of B. In the standard model both constructions lead to isomorphic objects, while this might not be the case in certain non-standard models. $(P(A) \times P(B) \cong P(A + B))$

Matrix Algebras

Definition

Let $\mathcal{R}$ be a relation algebra. The algebra $\mathcal{R}^+$ of the matrices with coefficients from $\mathcal{R}$ is defined by:

- The class of objects of $\mathcal{R}^+$ is the collection of all functions from an arbitrary set I to $Obj_{\mathcal{R}}$.
- For every pair $f : I \to Obj_{\mathcal{R}}, g : J \to Obj_{\mathcal{R}}$ of objects from $\mathcal{R}^+$, the set of morphisms $\mathcal{R}^+[f,g]$ is the set of all functions $R : I \times J \to Mor_{\mathcal{R}}$ such that $R(i,j) \in \mathcal{R}[f(i), g(j)]$ holds.
- For $R \in \mathcal{R}^+[f,g]$ and $S \in \mathcal{R}^+[g,h]$ composition is defined by $(R;S)(i,k) := \bigsqcup R(i,j); S(j,k)$.
- For $R \in \mathcal{R}^+[f,g]$ conversion and negation is defined by $R^{\smile}(j,i) := (R(i,j))^{\smile}$, $\overline{R}(i,j) = \overline{R(i,j)}$
- For $R, S \in \mathcal{R}^+[f,g]$ union and intersection is defined by $(R \sqcup S)(i,j) = R(i,j) \sqcup S(i,j)$, $(R \sqcap S)(i,j) = R(i,j) \sqcap S(i,j)$
- The identity, zero and universal elements are defined by

$$\mathbb{I}_{f(i_1,i_2)} := \begin{cases} \perp\!\!\!\perp_{f(i_1)f(i_2)} & : i_1 \neq i_2 \\ \mathbb{I}_{f(i_1)} & : i_1 = i_2 \end{cases}$$

$$\perp\!\!\!\perp_{fg(i,j)} := \perp\!\!\!\perp_{f(i)g(j)}, \quad \mathbb{T}_{fg(i,j)} := \mathbb{T}_{f(i)g(j)}$$

[3].

A Pseudo Representation Theorem

Lemma

$\mathfrak{R}^+$ is a relation algebra

Definition

An object A of a relation algebra is called integral iff $\perp\!\!\!\perp_{AA} \neq \top\!\!\!\top_{AA}$ and for all $Q, R \in \mathcal{R}[A, A]$, the equation $Q; R = \perp\!\!\!\perp_{AA}$ implies either $Q = \perp\!\!\!\perp_{AA}$ or $R = \perp\!\!\!\perp_{AA}$.

Definition

Let $\{A_i | i \in I\}$ be a set of objects indexed by a set I. An object $\sum_{i\in I} A_i$. together with relations $\iota_j \in \mathcal{R}[A_j, \sum_{i\in I} A_i]$ for all $j \in I$ is called a relational sum of $\{A_i | i \in I\}$ iff for all $i, j \in$ I with $i \neq j$ the following holds: $\iota_i;\breve{\iota_i} = \mathbb{I}_{A_i}$ $\quad \iota_i;\breve{\iota_j} = \perp\!\!\!\perp_{A_iA_j}$ $\quad \bigsqcup_{i\in I} \iota_i;\breve{\iota_i} = \mathbb{I}_{\sum_{i\in I} A_i}$

Theorem

Let $\mathcal{R}$ be a relation algebra with relational sums and subobjects and $\mathcal{B}$ be the basis of $\mathcal{R}$. Then $\mathcal{R}$ and $\mathcal{B}^+$ are equivalent [7].

Definition

Let $\Xi \in \mathcal{R}[A, A]$ be a partial equivalence relation. An object B together with a relation $\psi \in \mathcal{R}[B, A]$ is called a splitting of Ξ iff

$$\psi; \psi^{\smile} = \mathbb{I}_{\boldsymbol{B}}, \qquad \psi^{\smile}; \psi = \Xi.$$

A relation algebra has splittings iff for all partial equivalence relations a splitting exists.

Definition

Let $\mathcal{R}$ be a relation algebra. An object $\mathcal{P}(A)$, together with a relation $\varepsilon_{\boldsymbol{A}} : A \to \mathcal{P}(A)$ is called a relational power of A iff

$$\mathrm{syQ}(\varepsilon_{\boldsymbol{A}}, \varepsilon_{\boldsymbol{A}}) \sqsubseteq \mathbb{I}_{\mathcal{P}(\boldsymbol{A})} \quad \text{and} \quad \mathrm{syQ}(R, \varepsilon_{\boldsymbol{A}}) \text{ is total}$$

for all relations $R : B \to A$. $\mathcal{R}$ has relational powers iff the relational power for any object exists.

Theorem

If $\mathcal{R}$ is small, then $\mathcal{R}^+$ has pre-powers.

- We call an object B a pre-power of A if there is a relation $T : A \to B$ so that $\mathrm{syQ}(R, T)$ is total for all relations $R : C \to A$, i.e., a relational power is a pre-power with the additional requirement $\mathrm{syQ}(T, T) \sqsubseteq \mathbb{I}$. If $\mathcal{R}$ has splittings, then we obtain a relational power of A from a pre-power by splitting the equivalence relation $\mathrm{syQ}(T, T)$

Theorem

Let $\mathcal{R}$ be relation algebra with splittings. Then $\mathcal{R}^+$ has splittings.

Example

Let $M : f \to f$ be a partial equivalence relation in $\mathcal{R}^+$ of size α, and let $R_\beta : A_\beta \to f(\beta)$ be a splitting of Ξ_β in $\mathcal{R}$. Define $g : \alpha \to \mathrm{Obj}_{\mathcal{R}}$ and $N : g \to f$ by

$$g(\beta) = A_\beta,$$

$$N(\beta,\gamma) = \begin{cases} R_\beta ; M(\beta,\gamma) & \text{iff } \beta \leq \gamma, \\ \bot_{A_\beta f(\gamma)} & \text{otherwise,} \end{cases}$$

For our example we consider the object $[B_{\boldsymbol{abc}}, B_{\boldsymbol{abc}}, B_{\boldsymbol{abc}}, B_{\boldsymbol{abc}}]$ and following partial equivalence relation in matrix form:

$$M = \begin{array}{c} \\ B_{\boldsymbol{abc}} \\ B_{\boldsymbol{abc}} \\ B_{\boldsymbol{abc}} \\ B_{\boldsymbol{abc}} \end{array} \begin{array}{c} \begin{array}{cccc} B_{\boldsymbol{abc}} & B_{\boldsymbol{abc}} & B_{\boldsymbol{abc}} & B_{\boldsymbol{abc}} \end{array} \\ \begin{pmatrix} ab & 0 & b & 0 \\ 0 & ab & 0 & a \\ b & 0 & bc & 0 \\ 0 & a & 0 & a \end{pmatrix} \end{array}$$

Example

we obtain the following four partial equivalence relations $\Xi : B_{abc} \to B_{abc}$.

$$\begin{aligned}
\Xi_1 &:= ab, \\
\Xi_2 &:= ab \sqcap \overline{0 \sqcap 0} = ab \sqcap abc = ab, \\
\Xi_3 &:= bc \sqcap \overline{b \sqcap b} \sqcap \overline{0 \sqcap 0} = bc \sqcap ac \sqcap abc = c, \\
\Xi_4 &:= a \sqcap \overline{0 \sqcap 0} \sqcap \overline{a \sqcap a} \sqcap \overline{0 \sqcap 0} = a \sqcap abc \sqcap bc \sqcap abc = 0.
\end{aligned}$$

The splittings for each of those relations is given by

$$R_1 = R_2 := ab : B_{ab} \to B_{abc}, \quad R_3 := c : B_c \to B_{abc}, \quad R_4 := 0 : B_0 \to B_{abc}.$$

Notice that the source object of each of those relations is different from B_{abc}. We obtain the matrix N as:

$$N = \begin{array}{c} \\ B_{ab} \\ B_{ab} \\ B_c \\ B_0 \end{array} \begin{array}{c} \begin{array}{cccc} B_{abc} & B_{abc} & B_{abc} & B_{abc} \end{array} \\ \begin{pmatrix} ab & 0 & b & 0 \\ 0 & ab & 0 & a \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Objectives

- The proposed system will work with both standard and non-standard model.
- It will implement relation algebras in the form of matrix algebras.
- Make available a library to aid researchers to manipulate relations in their own programs.
- The proposed structure will be implemented in a packaged using C which can be imported into other programs and/or languages. A specific focus will be on a suitable module for the programming language Haskell.
- Understand how relations represented as matrices could be implemented using MDDs.

MDDs

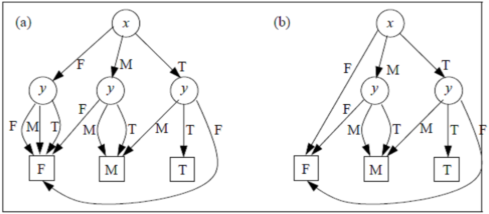
- A multiple-valued decision diagram (MDD) is a natural extension of the reduced ordered binary decision diagrams (ROBDDs) to the multiple-valued case
- MDDs are considered to be more efficient and require smaller memory size than the ROBDDs.
- Let V be a set of finite size r. An r-valued function f is a function mapping V^n for some n to V. We will identify the n input values of f using a set of variables $X = \{x_0, x_1, \ldots, x_n\}$. Each x_i as well as $f(X)$ is r-valued, i.e., it represents an element from V. The function f can be represented by a multi-valued decision diagram. Such a decision diagram is directed acyclic graph (DAG) with up to r terminal nodes each labeled by a distinct value from V. Every non-terminal node is labeled by an input variable x_i and has r outgoing edges
- An MDD is ordered (OMDD) if there is an order on the set of variables X so that for every path from the root to a leave node all variables appear in that order.
- MDD is called reduced if the graph does not contain isomorphic subgraphs and no nodes for which all r children are isomorphic (i.e. there is no redundant nodes in which two edges leaving a node points to the same next node within the graph).

Example MDDs

Let f be a multiple valued logic defined over the truth values T, M, F such that T = True, M = Maybe and F = False.

$\wedge$	F	M	T
F	F	F	F
M	F	M	M
T	F	M	T

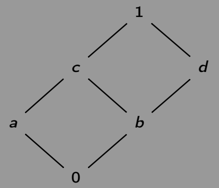
$\vee$	F	M	T
F	F	M	T
M	M	M	T
T	T	T	T



$f = y \wedge x$ (a)MDD; (b)ROMDD

Implementation

- Heyting algebra $\mathcal{L}$



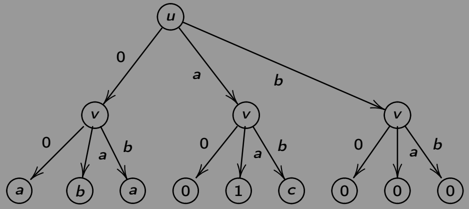
$$\begin{array}{c|cccccc} & 0 & a & b & c & d & 1 \\ \hline 0 & a & b & a & 0 & 0 & 0 \\ a & 0 & 1 & c & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

u	v	$f_R(u, v)$
0	0	a
0	a	b
0	b	a
a	0	0
a	a	1
a	b	c
_	_	0

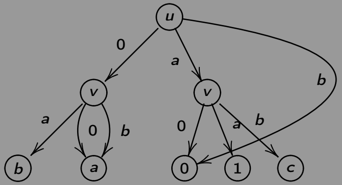
- Example: $c \sqcap d = b$, $a \sqcap b = 0$, $a \sqcap c = a$, $1 \sqcup b = 1$, and $a \sqcup b = c$
-

$$R = \begin{pmatrix} a & b & a \\ 0 & 1 & c \end{pmatrix} \quad S = \begin{pmatrix} a & a & 1 \\ 0 & b & d \end{pmatrix}$$

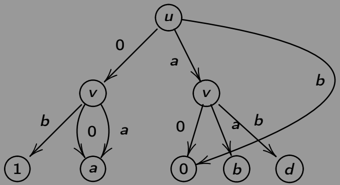
$$T := R \sqcap S = \begin{pmatrix} a & 0 & a \\ 0 & b & b \end{pmatrix}$$



Implementation



- Similarly, we construct the ROMDD for S:



- apply on ROMDDs: This operation applies a function to a leaf node of R and the corresponding leaf node of S. Corresponding leaf nodes are determined by the same path from the root to the leaf in both graphs.
- Another version of the apply operation applies a unary function to a each leaf node of a single ROMDD
- abstraction operation: This operation applies a function to all elements of an entire row or column of a matrix

RelMDD-A Library for Manipulating Relations Based on MDDs

- RelMDD is an arbitrary relation algebra manipulator library written in C programming language.
- It is a package that implements relation algebra using the matrix algebra approach and can be imported by other programs and/or languages such as Java and Haskell when programming or manipulating arbitrary relations.
- We implemented RelMDD using the matrix algebra over a suitable basis approach, hence it is capable of manipulating both the class of standard and the non-standard models of relation algebra.
- It has an XML interface for loading the basis of relations into memory.
- It has several function for performing basic operations.
- The implementation is currently restricted to the basic operations of relation algebras, i.e., union, intersection, composition, converse, and complement.
- In our implementation MDDs were implemented using algebraic decision diagrams [2]. By taking this approach we were able to use a well-known package for these diagrams called CUDD [5].

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- RELVIEW uses only standard procedures for OBDD manipulation.
- Relation algebra is made up of standard and non-standard models
- Relation algebra are Matrix algebra, hence we can implement relation algebra in a form of matrix algebra (not restricting ourselves to only boolean matrix.)
- In order to implement a complete system for relation algebra we used a more advance representation of functions using ROMDDs.

Future Work And Suggestions

- This work would be extended by creating a suitable module for the programming language Haskell or Java.
- Extending the this system into a programming language for relations
- Integrating the package into the RELVIEW System.
- A future project will add further operations such as sums and splittings.

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Thank You