Two Observations in Dioid Based Model Refinement

Roland Glück[1]

[1]Universität Augsburg

17.9.2012
Cambridge

About

- dioid-based optimality problems
- model refinement
- refinability
- bisimulations
- linear affine fixpoint equations

Recent Work

- recent work by Michel Sintzoff (MPC 2008)
- algebraic approach ar RelMiCS 2009
- generic algorithm at AMAST 2010
- predecessor paper at RAMiCS 2011

Basic Definition

Definition

A *complete dioid* is a structure $(D, \Sigma, 0, \cdot, 1)$ such that $(D, \sqsubseteq)$ is a complete lattice with supremum operator Σ and least element 0, where $\sqsubseteq$ is defined by $x \sqsubseteq y \Leftrightarrow \Sigma\{x, y\} = y$, $(D, \cdot, 1)$ is a monoid and $\cdot$ distributes over Σ from both sides. $\sqsubseteq$ is called the *order* of the complete dioid.

- naming dioid after Gondran/Minoux
- also known as quantale

Selective Dioids

- special case of *selective* dioids
- $a + b \in \{a, b\}$. i.e. $\sqsubseteq$ is linear
- abbreviation *s-dioid* for complete selective dioids
- e.g. $(\mathbb{R} \cup \{-\infty, \infty\}, \sup, -\infty, \inf, \infty)$, $(\mathbb{R} \cup \{-\infty, \infty\}, \inf, \infty, +, 0)$

Cumulative Dioids

- cumulative dioids
- characterised by $a \sqsubseteq 1$ for all $a \in D$
- equivalent to:
  - $\forall a, b, c \in D : a \sqsubseteq b \Rightarrow ac \sqsubseteq b \wedge a \sqsubseteq b \Rightarrow ca \sqsubseteq b$
  - $\forall a, b \in D : ab \sqsubseteq a \wedge ba \sqsubseteq a$
- interpretation will be given soon

Definition of Models

- model: pair (G, g) where
  - $G = (V, E)$ is a graph
  - $g : E \to D$ is an edge labelling function
  - D is carrier set of an s-dioid
- target model: model with target set $T \subseteq V$ (and some additional technical requirements)

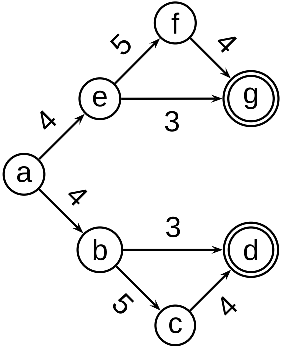
Costs in Models

- *cost* $c(w)$ of a walk $x_1 x_2 \ldots x_n$ in $M = (G, g)$ defined by $c(w) = \prod_{i=1}^{n-1} g(x_i, x_{i+1})$
- *distance* $d(x, y)$ by $d(x, y) = \sum_{w \in W(x,y)} c(w)$
- *target distance* in a target model by $d(x) = \sum_{t \in T} d(x, t)$
- $x_1 x_2 \ldots x_n$ is *optimal* walk if $c(x_1 x_2 \ldots x_n) = d(x_1, x_n)$
- not always existent

Interpretation

- suitable choices of D yield different optimality problems
- $(\mathbb{R} \cup \{-\infty, \infty\}, \text{inf}, \infty, +, 0)$ corresponds to shortest path problem
- $(\mathbb{R} \cup \{-\infty, \infty\}, \text{sup}, -\infty, \text{inf}, \infty)$ corresponds to maximum capacity path
- application in routing, planning, optimisation, ...

Example



target set

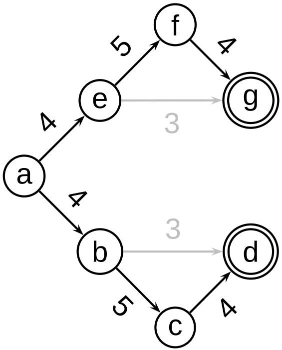
double surrounded

Optimal Submodels

- $((V, E'), g', T)$ *target submodel* of $((V, E), g, T)$ if $E' \subseteq E$ and $g' = g|_{E'}$
- $((V, E'), g', T)$ *optimal target submodel* of $((V, E), g, T)$ if
  - $((V, E'), g', T)$ is target submodel of $((V, E), g, T)$
  - all walks from arbitrary x into T are optimal
  - T is reachable from every node $x \in V - T$
- goal: *refine* given target model to an optimal target model

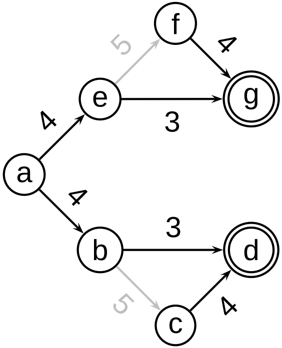
Refinement Algorithms

- in the case of finite node set:
- refinement in case of cumulative dioids by Dijkstra-like algorithm
- key point: prolonging a path can not improve its cost
- in general case by Floyd-Warshall-like algorithm
- in the absence of negative cycles



optimal submodel for

maximum capacity paths



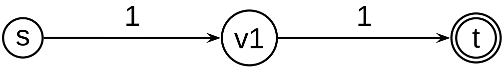
optimal submodel for

shortest paths

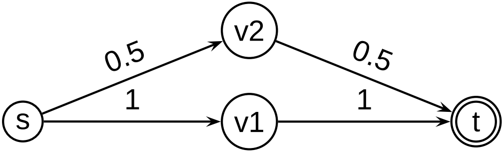
Refinability

- a model is called *refinable*, if it has an optimal submodel
- not every model is refinable
- negative cycles
- infinite node set

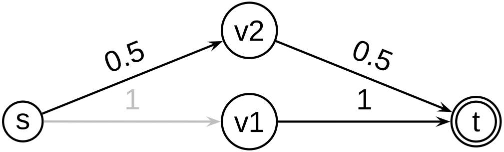
Infinite Carrier Set



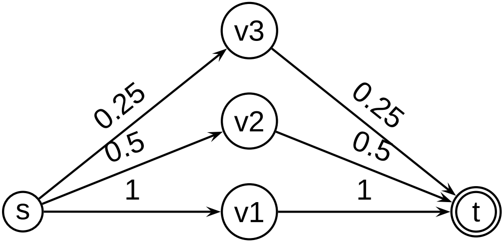
Infinite Carrier Set



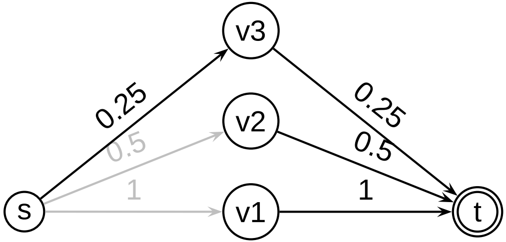
Infinite Carrier Set



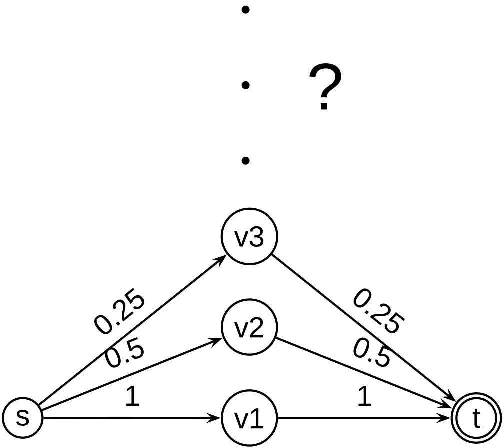
Infinite Carrier Set



Infinite Carrier Set



Infinite Carrier Set



Partial Result

Theorem

Every target model with labels drawn from a binary cumulative s-dioid is refinable.

- Conjecture: Every target model with edge labels from a finite cumulative s-dioid is refinable.

Definition

Definition

$B \subseteq V_1 \times V_2$ is a *bisimulation* between two graphs (V_1, E_1) and (V_2, E_2) iff

- $Dom(B) = X_1$ and $Cod(B) = X_2$
- $v_1 B v_2 \wedge v_1 E_1 w_1 \Rightarrow \exists w_2 : w_1 B w_2 \wedge v_2 E_2 w_2$
- $v_2 B^{\smile} v_1 \wedge v_2 E_2 w_2 \Rightarrow \exists w_1 : w_2 B^{\smile} w_1 \wedge v_1 E_1 w_1$

relational definition:

- $B^{\smile}; E_1 \subseteq E_2; B^{\smile} \ \wedge \ B; E_2 \subseteq E_1; B$

additional requirement: respecting edge labels

Coarsest Bisimulation

- bisimulations between G and itself are closed under
  - union,
  - composition, and
  - taking the converse
- identity is a bisimulation between G and itself
- existence of a *coarsest bisimulation equivalence on* G

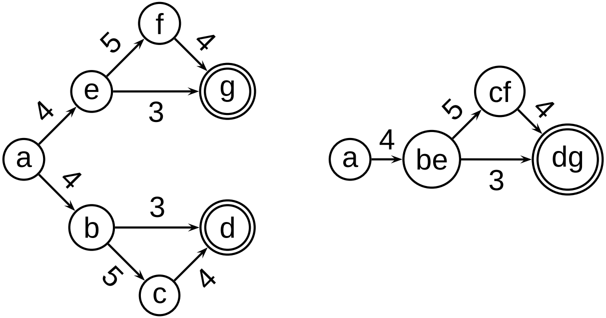
Compatible Bisimulations

- here main interest in bisimulations respecting $\{T, V - T\}$
- bisimulation equivalence B respects partition $V = \dot{\bigcup}_{i \in I} V_i$ if every V_i is the union of suitable equivalence classes of B
- for every partition of V there exists a coarsest respecting bisimulation

Quotient Graph

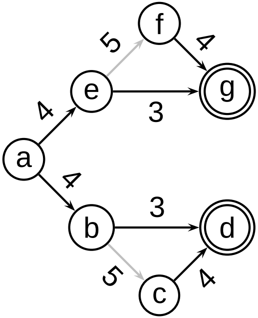
- for a bisimulation equivalence B and a graph $G = (V, E)$ the *quotient* $G/B = (V/B, E/B)$ is defined by
  - V/B is the set of equivalence classes of B
  - $(v/B, w/B) \in E/B \Leftrightarrow (v, w) \in E$
- G/B has in general a smaller node set then G
- *coarsest quotient* respecting T induced by coarsest bisimulation respecting $\{T, V - T\}$

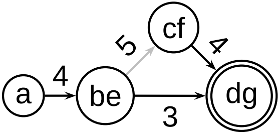
Example Quotient



- Algorithm Outline:
  - construct the coarsest quotient
  - refine the coarsest quotient
  - expand the solution to a solution of the original problem

Algorithm Example





Fixpoints and Optimal Paths

- optimal paths can be characterised as least fixpoint of a linear affine equation
- i.e. as least solution of $Ax + b = x$ (Bellman-Ford)
- A corresponds to adjacency matrix
- $b_v = 0$ for all $v \in V \backslash T$
- $b_v = 1$ for all $v \in T$
- addition and multiplication are drawn from dioid under consideration

Extension

- such equations can be considered for arbitrary b
- useful if worst case value is known a priori
- weighting of nodes in the target set

Result

Theorem

Linear affine fixpoint equations are compatible with taking the quotient.

- technical details are (as usual) annoying
- linear affine fixpoint can be computed via the quotient
- in certain cases runtime improvement

Outlook

- show refinability in the case of finite cumulative s-dioids
- idempotency is crucial point in many proofs
- extendable to more general algebraic structures?