Relational Representation Theorem for Powerset Quantales

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RAMiCS September 19, 2012

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Background: Quantales

- Def.
  A (unital) quantale is (Q,≦,∨,･,e) s.t.
  - (Q,≦,∨) is a complete join semilattice
  - (Q,･,e) is a monoid
  - (∨S)･a=∨{b･a | b∈S} for a∈Q, S⊆Q
  - a･(∨S)=∨{a･b | b∈S} for a∈Q, S⊆Q

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Background: An Example of Quantale

- (Rel(A),≦,∨,･,e) is a quantale where
  - Rel(A) is the set of all binary relations on A
  - ≦ is the inclusion
  - ∨ is the union
  - ･ is the composition
    Q･R={ (a,c) | ∃b. (a,b)∈Q, (b,c)∈R }
  - e is the identity relation

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Our Goal: Relational Representation Theorem

- Goal is
  - to give a sufficient condition for
    a quantale Q to be isomorphic to
    a subquantale of Rel(A) for some set A.
- In other words,
  - to give a sufficient condition for
    a quantale Q to have
    an injective homomorphism of quantales
    from Q to Rel(A) for some set A.

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Background: Related works

	The multiplication corresponds to composition of relations	The unit corresponds to the identity	The result is applicable to non-involutive quantales
Brown and Gurr, 1993	Yes	No	Yes
Valentini, 1994	Yes	No	Yes

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This paper	Yes	Yes	Yes

The Result of this paper

- For a quantale Q, the following are equivalent.
  1. Q is a powerset quantale.
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  2. Q has a relational representation in our way and it is CCP-invertible.

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  2. Q has a relational representation in our way and it is CCP-invertible.
     - 'CCP-invertible' is a notion defined in this paper.

Outline

1. Another characterization of powerset quantales
2. Relational representation theorem for powerset quantales
3. CCP-invertible quantales

Another characterization of powerset quantales

Another characterization of powerset quantales

- For a quantale,
  the following are equivalent.
  1. Its underlying complete join semilattice
     is isomorphic to the powerset of some set.
  2. Its underlying complete join semilattice
     is CCP-algebraic and
     its order of CCP elements is discrete.

Complete join semilattice

- Def.
  A complete join semilattice is $(L, \leqq, \vee)$ s.t.
  - $(L, \leqq)$ is a partially ordered set
  - $\vee S$ is the join (the least upper bound)
    for each $S \subseteq L$
    - Note: $\perp$ is given by $\vee \phi$

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- E.g. P(A) ... powerset of a set A
- E.g. $Rel(A) = P(A \times A)$
- E.g. $N \cup \{\omega\}$
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CCP elements

- Def.
  An element x of a complete join semilattice $(L, \leqq, \vee)$
  is called CCP (Completely CoPrime), if for any $S \subseteq L$,

  $x \leqq \vee S \Leftrightarrow \exists a \in S.\ x \leqq a.$

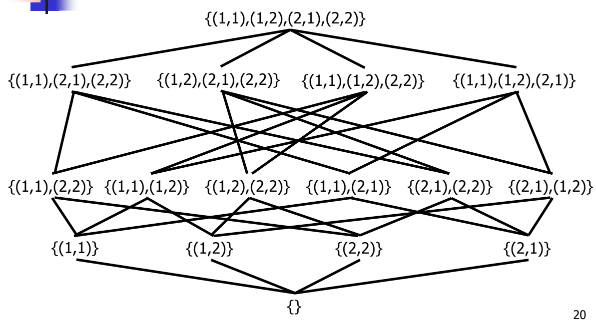
- E.g. CCP elements of P(A) or Rel(A) are singletons.
  - If $X = \phi$, then $X \subseteq \cup \phi$ but not $\exists Y \in \phi.\ X \subseteq Y$
  - If $X = \{a\}$, then $\{a\} \subseteq \cup S \Leftrightarrow a \in \cup S \Leftrightarrow \exists Y \in S.\ a \in Y$
  - If $X = \{a, b, \ldots\}$, then $X \subseteq \{a\} \cup \{b, \ldots\}$
    but neither $X \subseteq \{a\}$ nor $X \subseteq \{b, \ldots\}$

CCP-algebraic

- Def.
  A complete join semilattice (L,≦,∨) is called CCP-algebraic, if for any a∈L,

  a=∨{ x ｜ x is CCP, x≦a}.

E.g. Rel({1,2})



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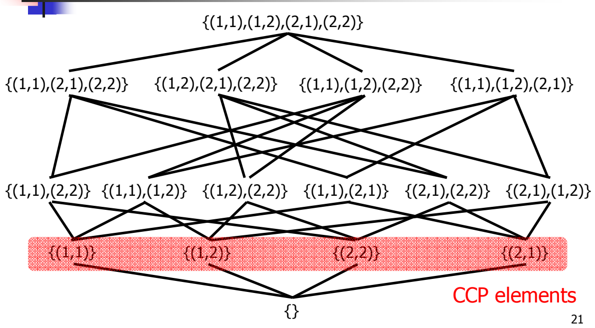

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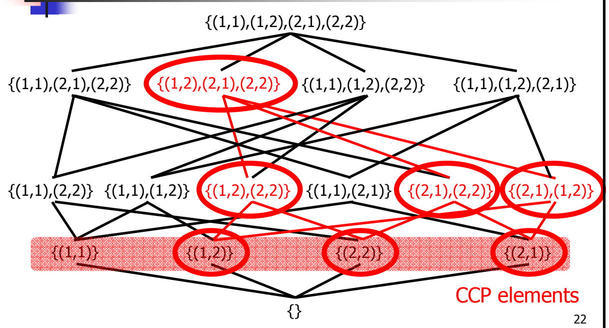
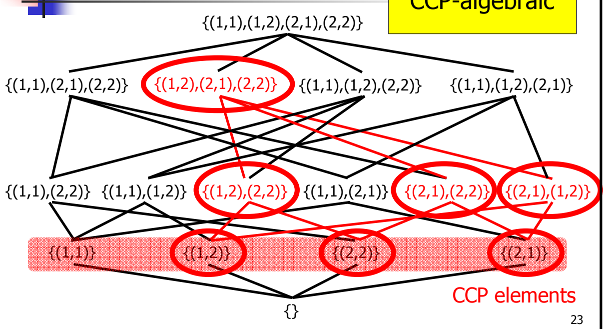


E.g. Rel({1,2})

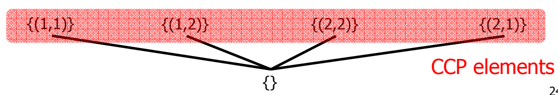
Rel({1,2}) is CCP-algebraic



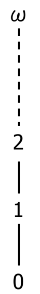
E.g. Rel({1,2})

Rel({1,2}) is CCP-algebraic

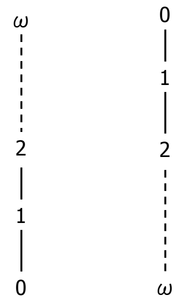
The order of the CCP elements is discrete.
(If CCP elements x,y satisfy x≦y, then x=y.)



E.g. $N \cup \{\omega\}$



E.g. $N \cup \{\omega\}$ and $(N \cup \{\omega\})^{op}$



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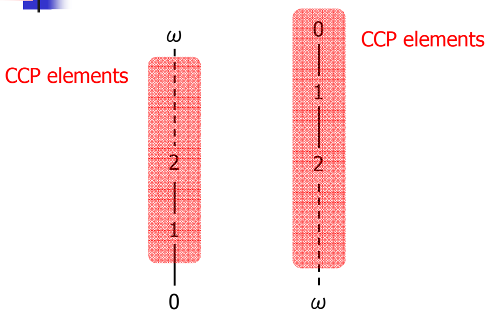

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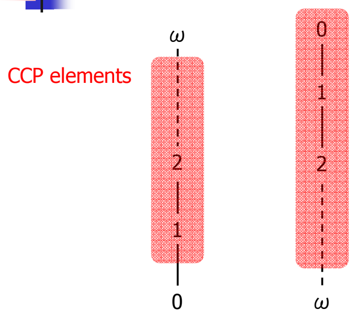

Both are CCP-algebraic

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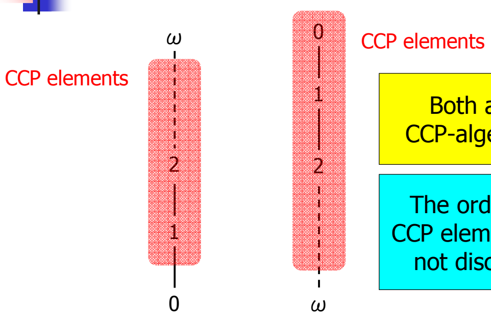
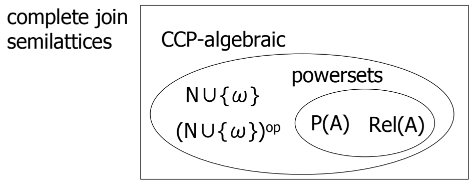


Both are CCP-algebraic

The order of CCP elements is not discrete.

Other characterizations of powerset lattices

- For a complete join semilattice Q, the following are equivalent.
  1. Q is isomorphic to the powerset of some set.
  2. Q is CCP-algebraic and the order of its CCP elements is discrete.



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  1. Q is isomorphic to the powerset of some set.
  2. Q is CCP-algebraic and the order of its CCP elements is discrete.

Other characterizations of powerset lattices

- For a complete join semilattice Q, the following are equivalent.
  1. Q is isomorphic to the powerset of some set.
  2. Q is CCP-algebraic and the order of its CCP elements is discrete.
  3. Q is CCP-algebraic and its CCP elements are atoms
  4. Q is atom-algebraic and its atoms are CCP
  5. Q is atom-algebraic and it is a frame

(The proof is shown in the paper.)

Relational Representation Theorem for Powerset Quantales

Powerset quantales

- Def. Q is called a powerset quantale, if its underlying complete join semilattice is isomorphic to the powerset of some set.
- E.g. The set of languages
  - $(P(A^*), \subseteq, \cup, \cdot, \{\varepsilon\})$
- E.g. The set of relations
  - $(Rel(A), \subseteq, \cup, \cdot, id)$
- E.g. Every powerset with $\cdot = \cap$
  - $(P(A), \subseteq, \cup, \cap, A)$
  - $(P(A^*), \subseteq, \cup, \cap, A^*)$
  - $(Rel(A), \subseteq, \cup, \cap, A \times A)$

Main Theorem

- If $(Q, \leqq, \vee, \cdot, e)$ is a powerset quantale, then the following η is an injective homomorphism of quantales.

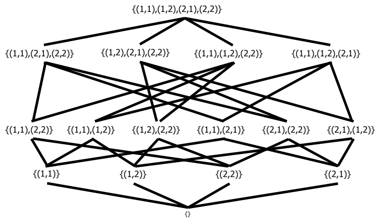
  $\eta : Q \to Rel(CCP(Q))$
  $\eta(a) = \{ (x,y) \mid x,y \in CCP(Q),\ x \leqq a \cdot y \}$

  where $CCP(Q) = \{x \in Q \mid x \text{ is CCP}\}$

Outline of the proof

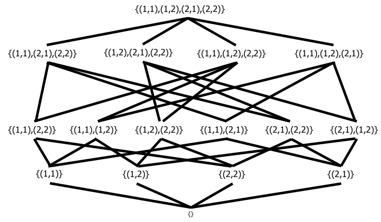
- Since Q is a powerset quantale,
  (Assumption 1) Q is CCP-algebraic and
  (Assumption 2) the order of CCP elements is discrete.
- $\eta : Q \to Rel(CCP(Q))$
  $\eta(a) = \{ (x,y) \mid x,y \in CCP(Q),\ x \leqq a \cdot y \}$
  - maps $\vee$ to the union (proved by Def. of CCP)
  - maps $\cdot$ to composition (proved by Assumption 1)
  - maps e to the identity (proved by Assumption 2)
  - is injective (proved by Assumption 1)

E.g. Representation of Rel({1,2})



E.g. Representation of Rel({1,2})

η : Rel({1,2}) → Rel(CCP(Rel({1,2})))



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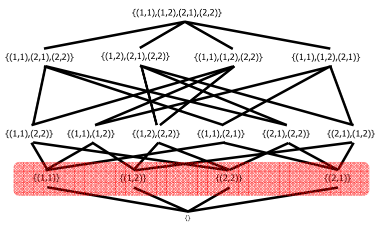

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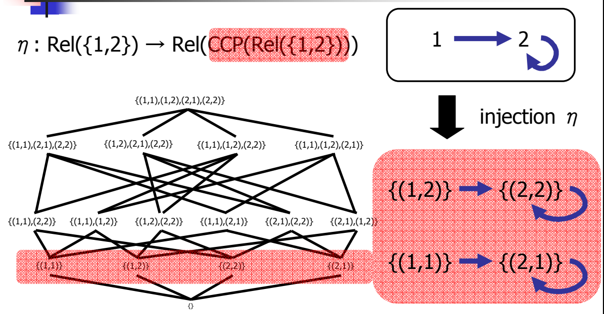

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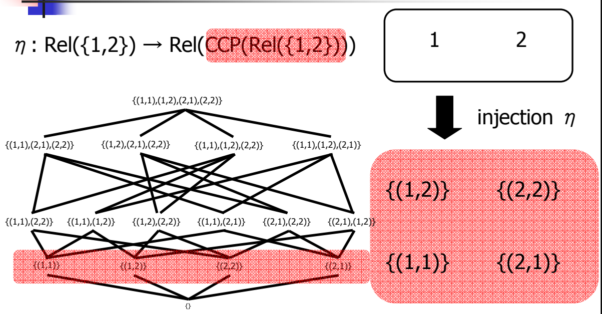

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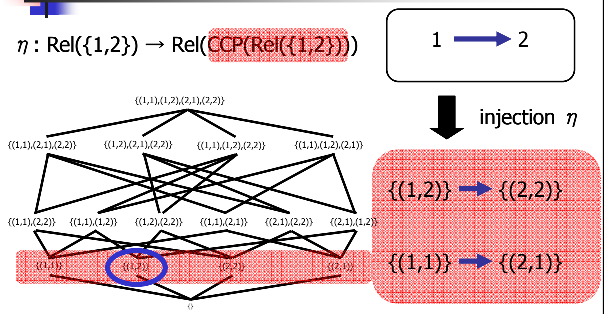

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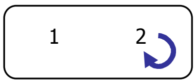

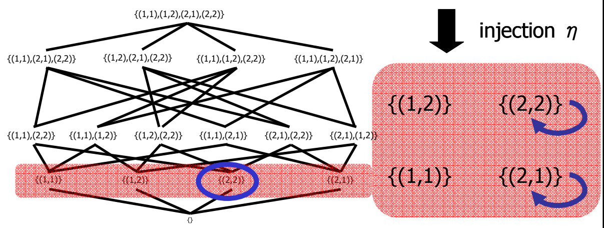

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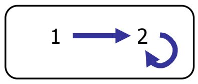

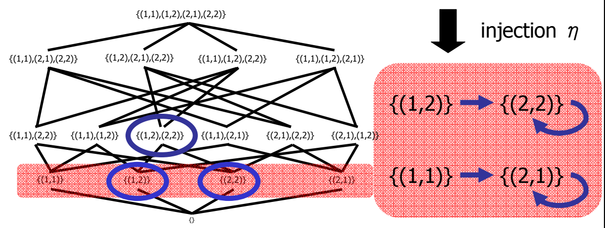

CPP-invertible Quantales

Is the powerset condition necessary and sufficient ?

- If (Q,≦,∨,・,e) is a powerset quantale, then η is an injective homomorphism of quantales.
- Is it the necessary and sufficient condition ?
- No.

Extension of The Representation Theorem

- For a quantale Q, the following are equivalent.
  1. Q is a powerset quantale.
  2. Q has a relational representation in our way and it is CCP-invertible.

CCP-invertible Quantale

- Def.
  A quantale (Q,≦,∨,・,1) is called CCP-invertible, if for any a∈Q, x,y∈CCP(Q), it holds that

  x≦a・y ⇔ ∃z∈CCP(Q). z≦a and x≦z・y.

CCP-invertible Quantale

- Def.
  A quantale $(Q, \leqq, \vee, \cdot, 1)$ is called <u>CCP-invertible</u>, if for any $a \in Q$, $x, y \in CCP(Q)$, it holds that

  $x \leqq a \cdot y \Leftrightarrow \exists z \in CCP(Q).\ z \leqq a$ and $x \leqq z \cdot y$.
- e.g. Rel(A)
  - If $x \leqq a \cdot y$, then the above z is given by $x \cdot y\#$ where $y\#$ is the converse relation of y.

CCP-invertible Quantale

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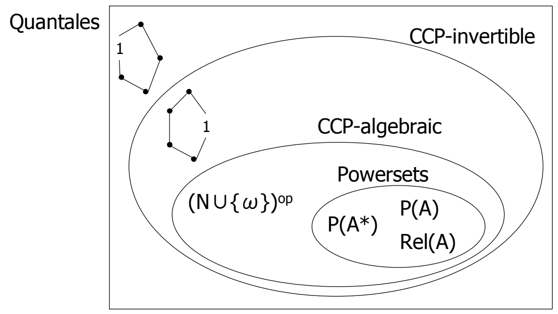
- e.g. Rel(A)
  - If $x \leqq a \cdot y$, then the above z is given by $x \cdot y\#$ where $y\#$ is the converse relation of y.
- e.g. $((N \cup \{\omega\})^{op}, \min, \omega, +, 0)$
  - If $x \leqq a \cdot y = a + y$, then the above z is given by $x - y$.

Other examples



Extension of The Representation Theorem

- For a quantale Q, the following are equivalent.
  1. Q is a powerset quantale.
  2. Q has a relational representation in our way and it is <u>CCP-invertible</u>.

  (The proof is shown in the paper.)

Extension of The Representation Theorem

- For a quantale Q, the following are equivalent.
  1. Q is a powerset quantale.
  2. Q has a relational representation in our way and it is <u>CCP-invertible</u>.

  (The proof is shown in the paper.)

In other words,

- For a <u>CCP-invertible</u> quantale Q, the following are equivalent.
  1. Q is a powerset quantale.
  2. Q has a relational representation in our way.

Conclusion

- For a quantale Q, the following are equivalent.
  1. Q is a powerset quantale.
  2. Q has a relational representation in our way and it is CCP-invertible.
- The 1st condition depends only on its underlying complete join semilattice.
- It is future work to
  - give a sufficient condition for a quantale to have a relational representation in <u>some</u> way.
  - give another characterization of CCP-invertible quantales.

Thank you