Relations on Hypergraphs

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Relations on a Set

Boolean algebra

Converse

$$\smallsmile\smallsmile R = R$$

Complement

$$- - R = R$$

Composition & residuation

$\bigcup$-preserving fns
on lattice of subsets
(a Boolean algebra)

Relations on a Hypergraph

Bi-Heyting algebra

Adjoint pair of converses

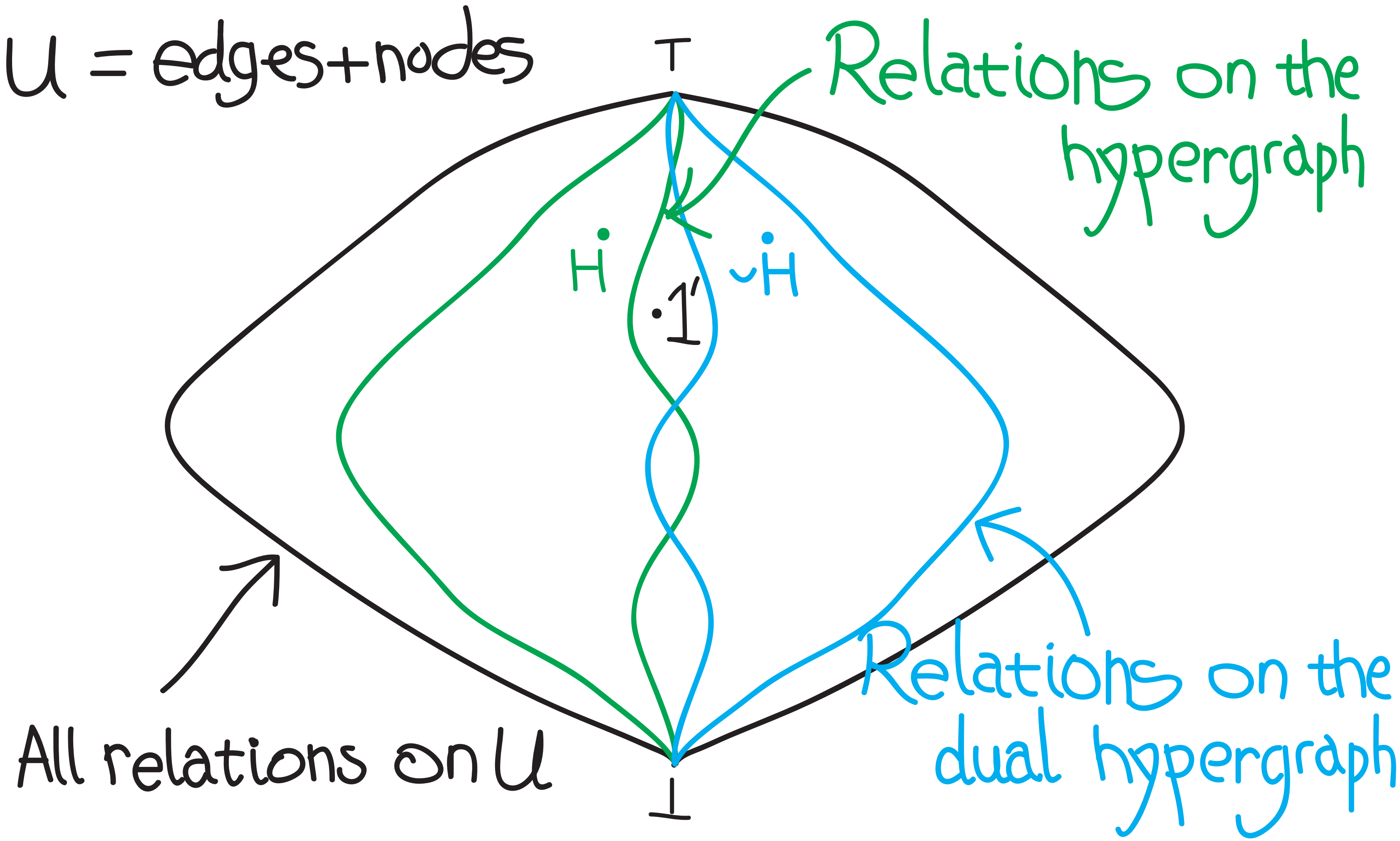
$$\curvearrowright\curvearrowleft R \subseteq R \subseteq \curvearrowleft\curvearrowright R$$

Pseudocomplement and dual

$$\lrcorner\lrcorner \subseteq R \subseteq \neg\neg R$$

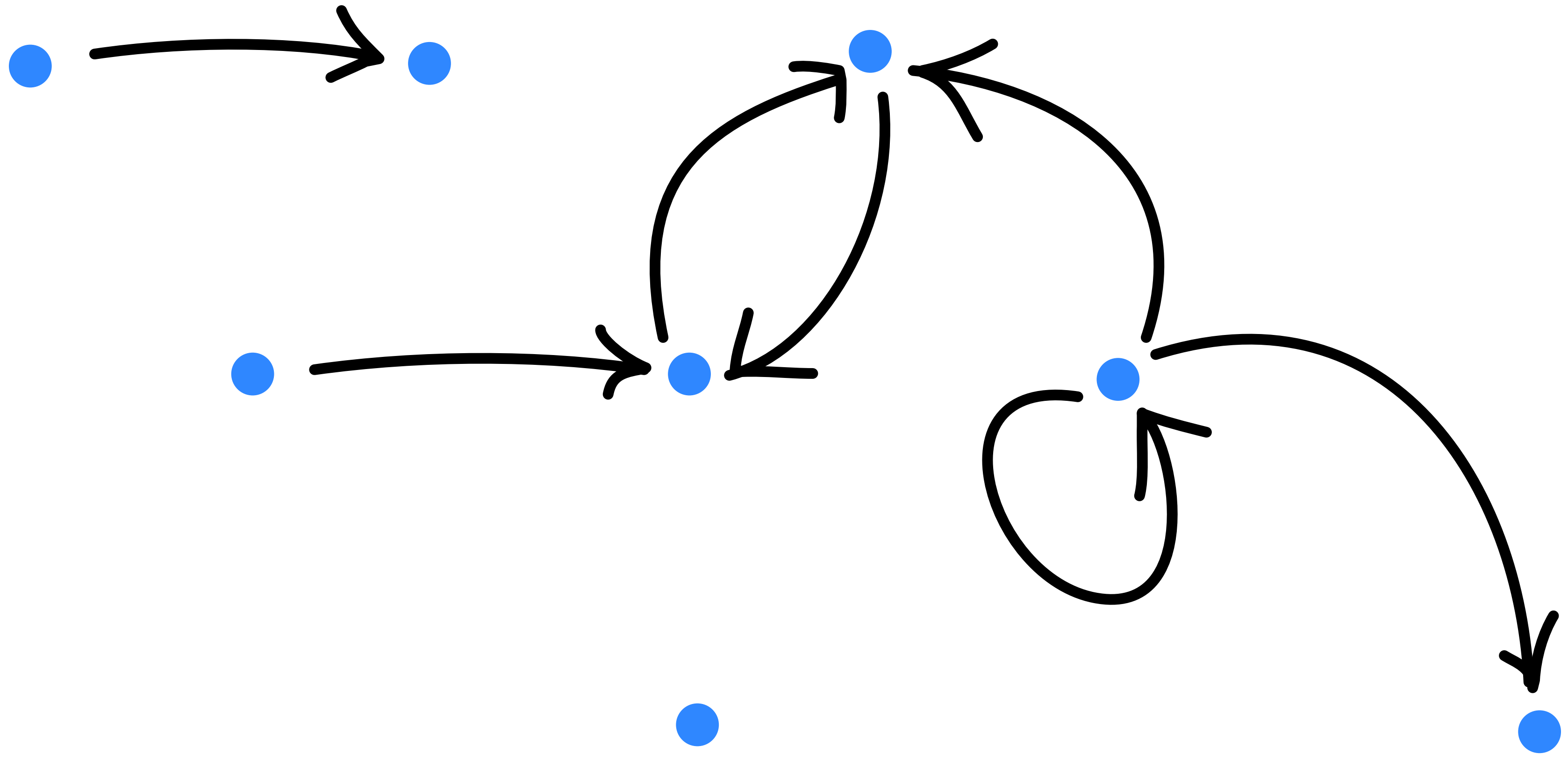
Composition & residuation

$\bigcup$-preserving fns
on lattice of subhypergraphs
(a bi-Heyting algebra)



U = edges + nodes
$\top$
Relations on the hypergraph
$\dot{H}$
$\cdot 1'$
${}^{\smile}\dot{H}$
All relations on U
$\perp$
Relations on the dual hypergraph

we are familiar with relations on sets, which we visualize (next slide) as arrows between dots . . .



a relation on a graph has arrows that can link

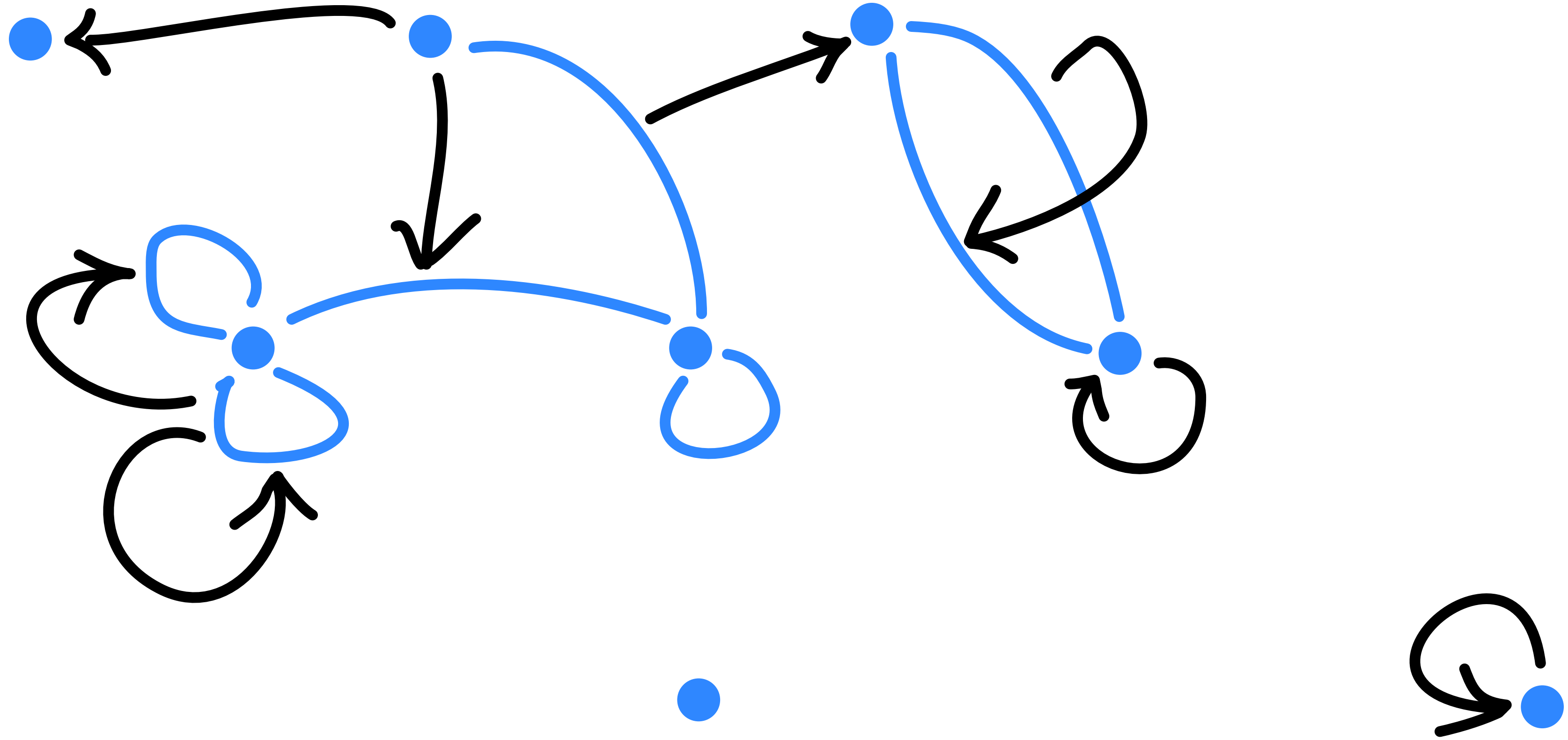
edges to edges

edges to nodes

nodes to edges

nodes to nodes

as in the next slide . . .

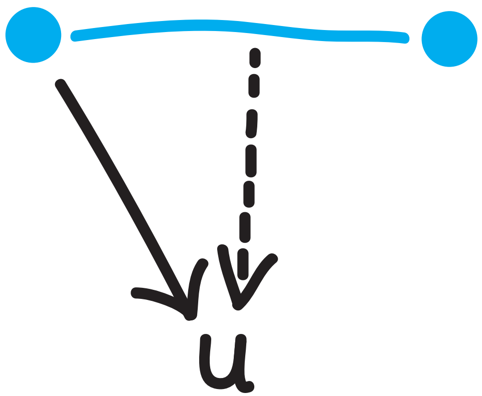


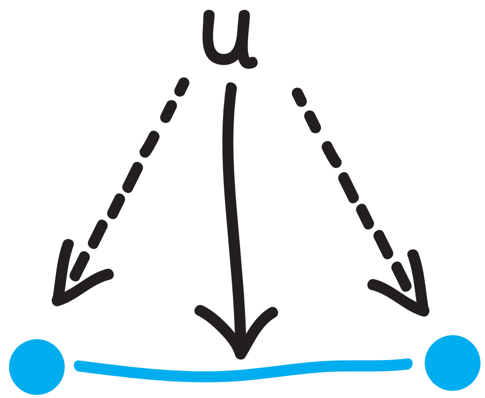
but we need the relation to interact nicely with the structure of the graph . . .

To express the condition we need on the relation R,
let u be any edge or node, n be any node and e any edge.

If $n\,R\,u$ then every edge incident with n is also related to u.

If $u\,R\,e$ then u is also related to every node incident with e.



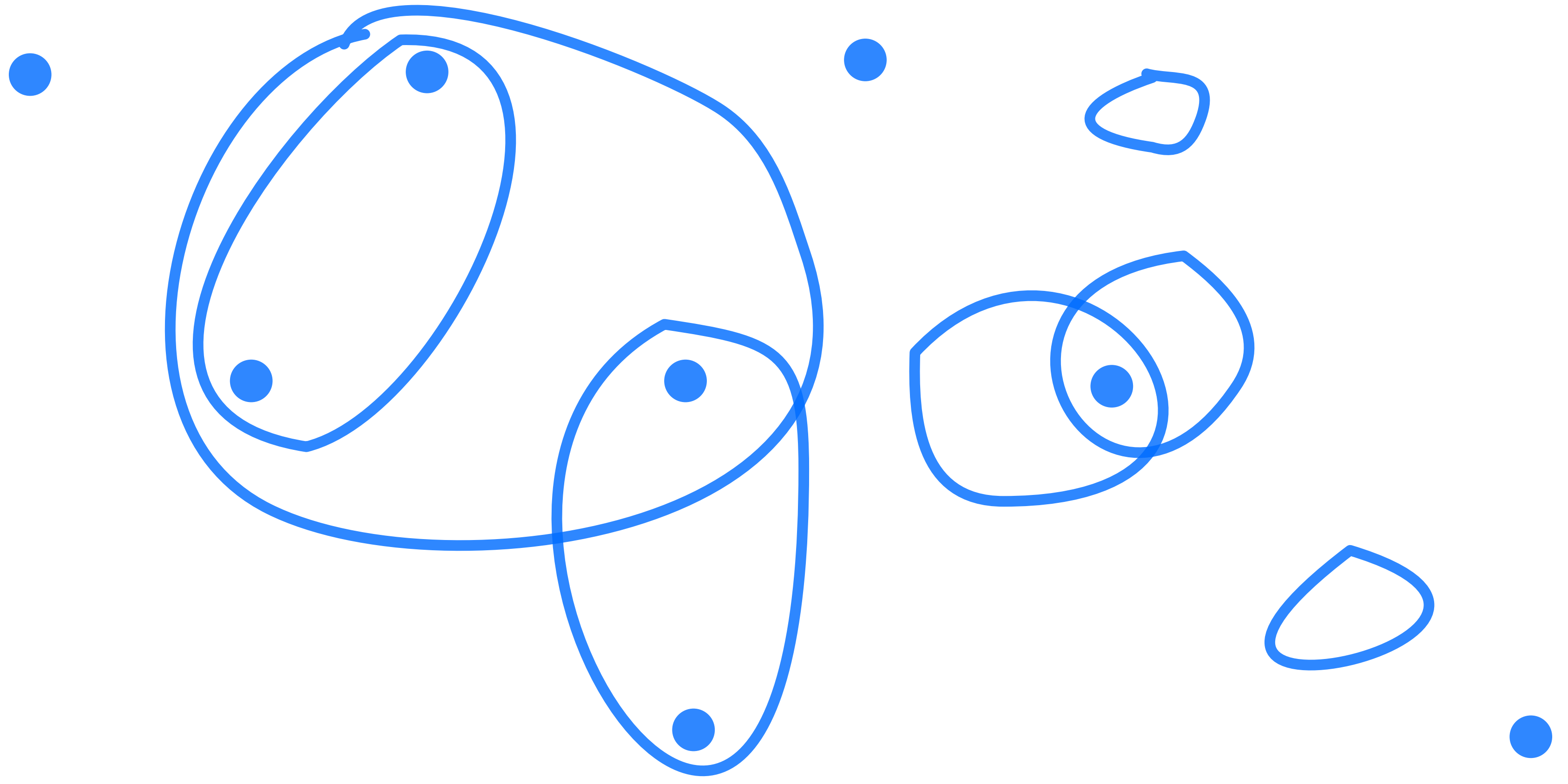


It's better to work with graphs rather than hypergraphs because of the edge-node duality they have.

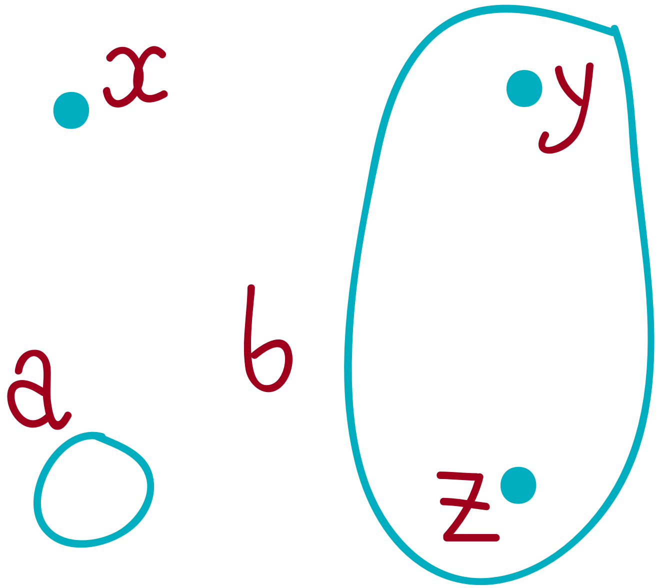
A **Hypergraph** consists of a set N of nodes and E of edges and an incidence relation associating edges to sets of nodes.

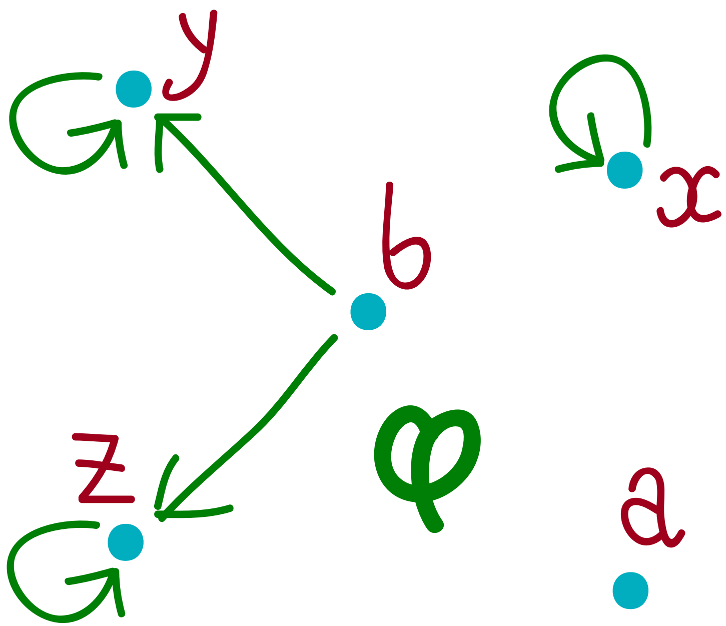
An edge may be incident with no edges and several edges may be incident with the same set of nodes.

A sub-hypergraph is a subset of the edges and nodes such that when any edge is included all its incident nodes are included.



Two views of the same hypergraph:





Hypergraphs with nodes N and edges E are equivalent to (binary) relations φ on $U = N \cup E$ such that

1. if $(x, y) \in \varphi$ then $(y, y) \in R$, and

2. if $(x, y) \in \varphi$ and $(y, z) \in R$ then $y = z$.

Given such a relation we can re-capture E and N as

$$E = \{u \in U : (u, u) \notin \varphi\}$$

$$N = \{u \in U : (u, u) \in \varphi\}$$

Define a relation R on a hypergraph (U, φ) to be $R \subseteq U \times U$ such that

$$\varphi \,;\, R \subseteq R \text{ and } R \,;\, \varphi \subseteq R.$$

Thm These relations correspond to the join-preserving functions on the lattice of sub-hypergraphs.

Propn R is a relation on (U, φ) iff $R = (\varphi \cup 1') \,;\, R \,;\, (\varphi \cup 1')$.

Defn Let H be a pre-order on U. Then $R \subseteq U \times U$ is an H-relation if $R = H \,;\, R \,;\, H$.

Basic properties of H-relations

Write Rel for poset of all relations on U and H-Rel for the poset of H-relations.

H-Rel is closed under composition, with identity H.

The inclusion H-Rel $\subseteq$ Rel has adjoints as follows.

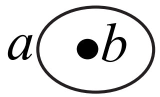
$$\begin{tikzcd}[column sep=huge] \mathsf{Rel} \arrow[r, shift left=2, "H \,;\, \_ \,;\, H"] \arrow[r, phantom, "\perp"] & H\text{-}\mathsf{Rel} \arrow[l, hook', shift left=2] \arrow[r, hook, shift left=2] \arrow[r, phantom, "\perp"] & \mathsf{Rel} \arrow[l, shift left=2, "H \setminus \_ \,/\, H"] \end{tikzcd}$$

where $f \dashv g$ means f is left adjoint to g.

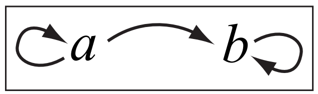
Hence, H-Rel is closed under arbitrary unions and intersections and includes $1 = U \times U$ and $0 = \varnothing$.

H-Rel is not closed under converse or complement, for example:

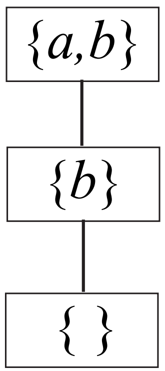
Hypergraph

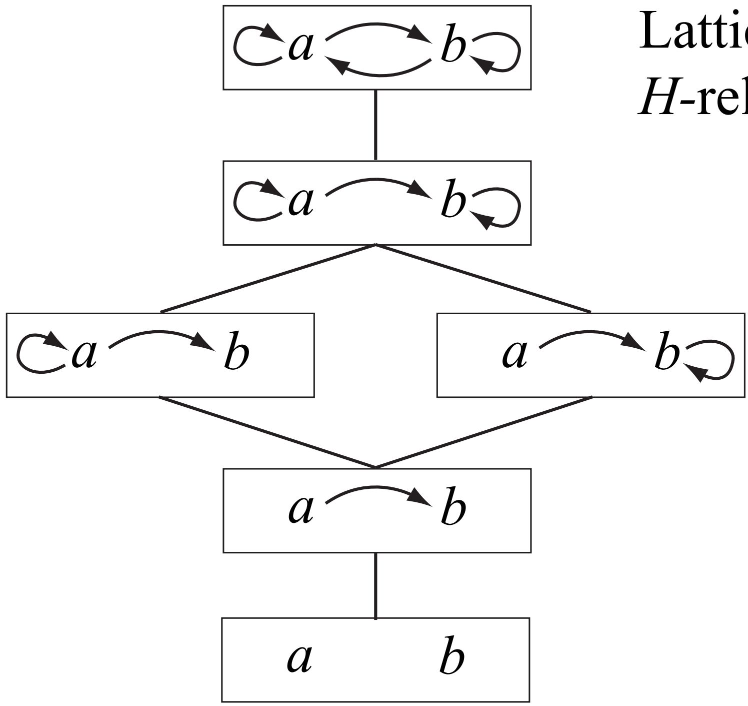


Relation H



Lattice of subgraphs





Lattice of H-relations

Denote converse of R by $\smile R$, and complement of R by $-R$, and use $\frown R$ to denote $\smile - R = -\smile R$.

If H comes from a hypergraph, then the $(\smile H)$-relations are the relations on the dual hypergraph.

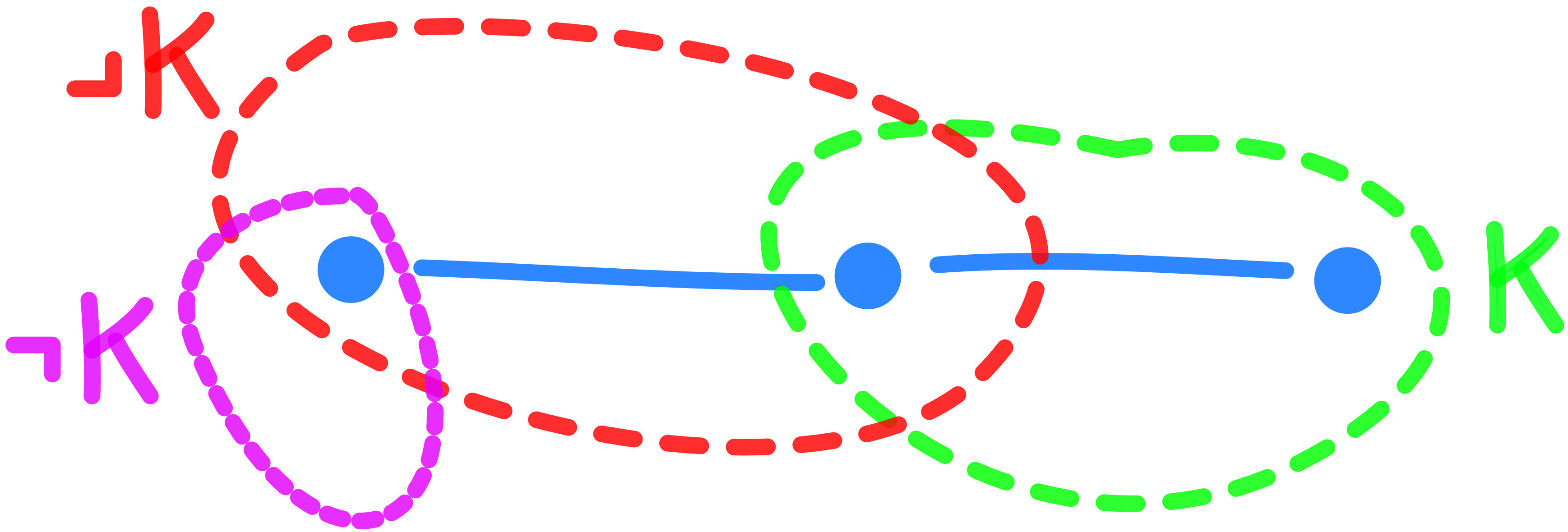
For $A \in \mathsf{Rel}$ the following four statements are equivalent

$$
\begin{array}{rrcl}
\text{(i)} & A & \in & H\text{-Rel}, \\
\text{(ii)} & -A & \in & \smile H\text{-Rel}, \\
\text{(iii)} & \smile A & \in & \smile H\text{-Rel}, \\
\text{(iv)} & \frown A & \in & H\text{-Rel}.
\end{array}
$$

$R \mapsto \frown R$ is an isomorphism of posets H-Rel $\to$ $(H\text{-Rel})^{\mathsf{op}}$.

Recall that the lattice of subgraphs of a graph is a bi-Heyting algebra.

In particular the Boolean complement of subsets becomes two weaker operations when we move to subgraphs (or subhypergraphs)



Generalizations of complement in H-Rel

In H-Rel we can define

$$\begin{aligned} \neg R &= H \setminus -R \,/\, H \\ \lrcorner R &= H \,;\, -R \,;\, H \end{aligned}$$

$\neg$ is a pseudocomplement and $\lrcorner$ a dual pseudocomplement.

More generally we get a relative pseudocomplement and a dual relative pseudocomplement

$$\begin{aligned} R \Rightarrow S &= H \setminus (-R \cup S) \,/\, H \\ S \searrow R &= H \,;\, (S \cap -R) \,;\, H \end{aligned}$$

<u>**Thm**</u> H-Rel is a complete bi-Heyting algebra which is isomorphic to its opposite and which is also isomorphic to the lattice of $\smile H$-relations.

Generalizations of converse in H-Rel

In H-Rel we can define

$$\circlearrowleft R \;=\; H \setminus {\smile}R \,/\, H \qquad \text{right converse}$$

$$\circlearrowright R \;=\; H \,;\, {\smile}R \,;\, H \qquad \text{left converse}$$

What happens to these familiar properties?

$${\smile}{\smile}R = R, \quad {\smile}(R \,;\, S) = {\smile}S \,;\, {\smile}R, \quad {\smile}1' = 1'$$

It is straightforward to construct relations R and S where

$$R \subsetneq \circlearrowright^2 R \subsetneq \circlearrowright^4 R \subsetneq \circlearrowright^6 R \subsetneq \circlearrowright^8 R \subsetneq \cdots,$$

and

$$\cdots \circlearrowleft^8 S \subsetneq \circlearrowleft^6 S \subsetneq \circlearrowleft^4 S \subsetneq \circlearrowleft^2 S \subsetneq \circlearrowleft S.$$

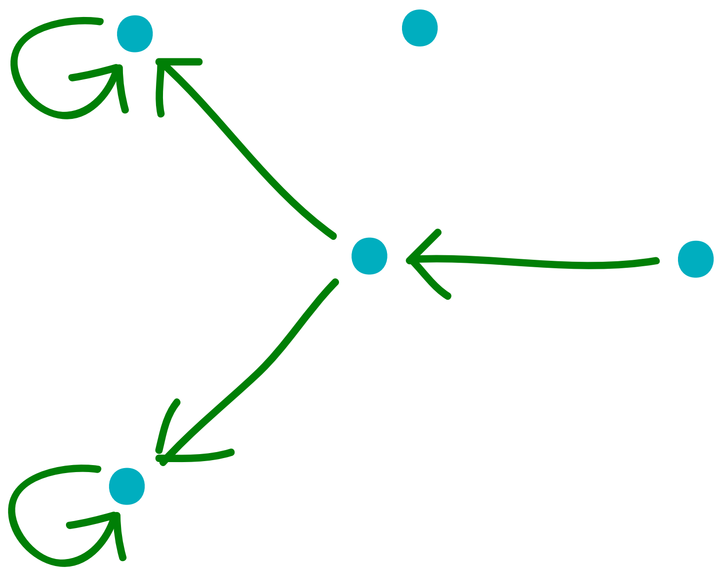
The left and right converses are adjoints.
$$\begin{tikzcd} H\text{-Rel} \arrow[r, shift left=2, "\circlearrowleft"] \arrow[r, phantom, "\perp"] & H\text{-Rel} \arrow[l, shift left=2, "\circlearrowright"] \end{tikzcd}$$

The following identities hold for all $R, S \in H$-Rel,

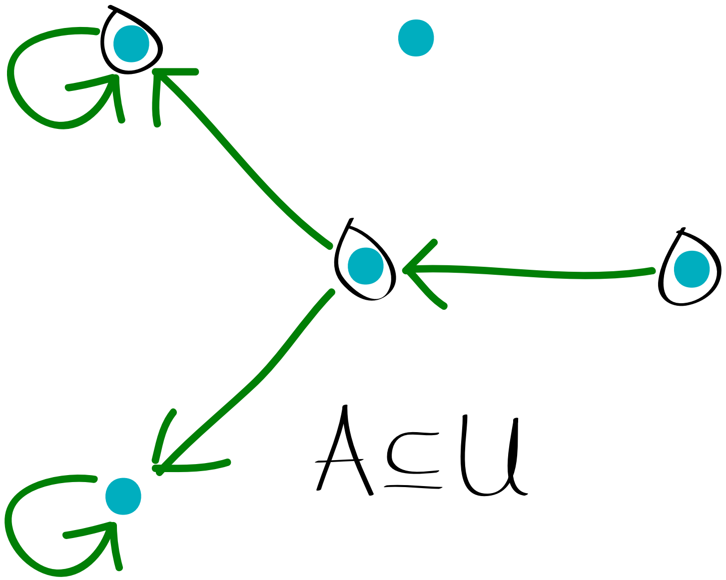
$$\begin{array}{rclcrclcrcl}
\neg R &=& \frown\circlearrowleft R & & \circlearrowleft R &=& \frown\neg R & & \lrcorner\neg R &=& \circlearrowleft\circlearrowleft R \\
\neg R &=& \circlearrowright\frown R & & \circlearrowright R &=& \neg\frown R & & \neg\lrcorner R &=& \circlearrowright\circlearrowright R \\
\lrcorner R &=& \frown\circlearrowright R & & \circlearrowright R &=& \frown\lrcorner R & & \neg\neg R &=& \circlearrowright\circlearrowleft R \\
\lrcorner R &=& \circlearrowleft\frown R & & \circlearrowleft R &=& \lrcorner\frown R & & \lrcorner\lrcorner R &=& \circlearrowleft\circlearrowright R
\end{array}$$

$$\circlearrowleft(R \cup S) = \circlearrowleft R \cup \circlearrowleft S, \qquad \circlearrowleft(R \cap S) = \lrcorner\lrcorner(\circlearrowleft R \cap \circlearrowleft S),$$
$$\circlearrowright(R \cap S) = \circlearrowright R \cap \circlearrowright S, \qquad \circlearrowright(R \cup S) = \neg\neg(\circlearrowright R \cup \circlearrowright S).$$

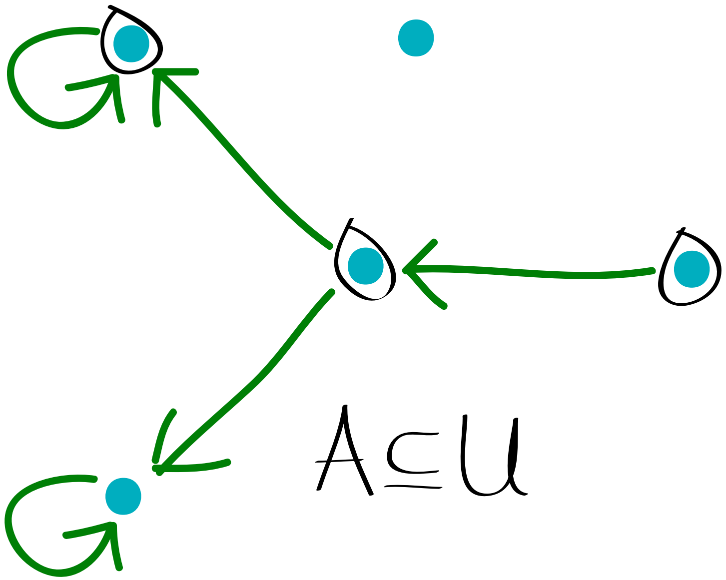
recall how relations on a set act on subsets. This is the relation

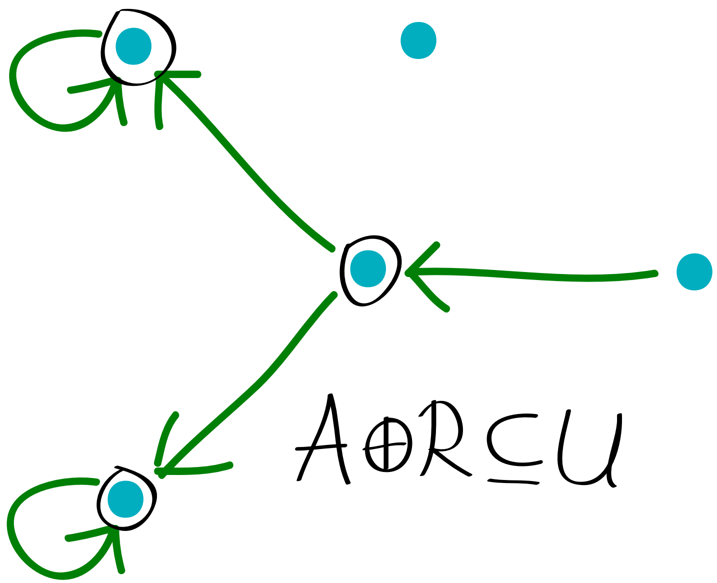


and here's the subset

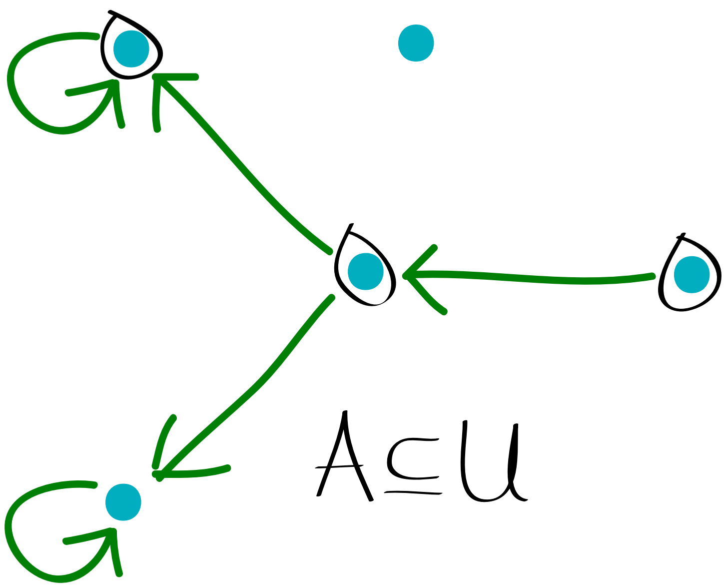


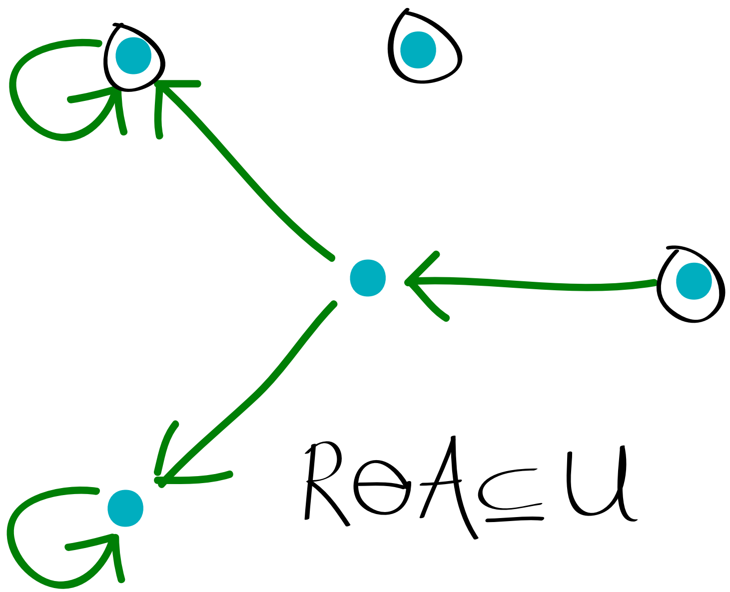
which has a ‘dilation’:





and an ‘erosion’:





dilation by the converse has this property

$$A \oplus \cup R = -(R \ominus -A)$$

which is a special case of operations on relations

$$A \oplus {\smile}R = -(R \ominus -A)$$

$$S ; {\smile}R = -(-S / R)$$

so we don't really need this

$$\cancel{A \oplus \smile R = -(R \ominus -A)}$$

$$S ; \smile R = -(-S / R)$$

but for relations acting on subsets, it’s not so simple. We have:

$$K \oplus \curvearrowright R = \neg(R \ominus \neg K)$$

and we might expect this one as well

$$K \oplus {}^{\smile}R = \neg(R \ominus \neg K)$$

$$S \,;\, {}^{\smile}R = \neg(\neg S / R)$$

but no.

$$K \oplus \breve{R} = \neg(R \ominus \neg K)$$

$$\cancel{S ; \breve{R} = \neg(\neg S / R)}$$

The theory of mathematical morphology for graphs is at an early stage.

Relations on a Set	Relations on a Hypergraph
Boolean algebra	Bi-Heyting algebra
Converse $\smile\smile R = R$	Adjoint pair of converses $\circlearrowleft\circlearrowright R \subseteq R \subseteq \circlearrowright\circlearrowleft R$
Complement $- - R = R$	Pseudocomplement and dual $\lrcorner\lrcorner \subseteq R \subseteq \neg\neg R$
Composition & residuation	Composition & residuation
$\bigcup$-preserving fns on lattice of subsets (a Boolean algebra)	$\bigcup$-preserving fns on lattice of subhypergraphs (a bi-Heyting algebra)