Point Axioms in Dedekind Categories

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RAMiCS 13 @ Cambridge
20th. September, 2012

Dedekind Category

A *Dedekind category* (locally complete division allegory) $\mathcal{D}$ is a category satisfying the following:

- $(\mathcal{D}(X,Y), \sqsubseteq, \sqcap, \sqcup, \Rightarrow, 0_{XY}, \nabla_{XY})$ is a complete Heyting algebra
- Converse $^\sharp : \mathcal{D}(X,Y) \to \mathcal{D}(Y,X)$ satisfying
  $(\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp$, $(\alpha^\sharp)^\sharp = \alpha$, $\alpha^\sharp \sqsubseteq \alpha'^\sharp$ if $\alpha \sqsubseteq \alpha'$
  is given

For $\alpha : X \rightharpoonup Y$, $\beta : Y \rightharpoonup Z$, $\gamma : X \rightharpoonup Z$

- Dedekind formula $\alpha\beta \sqcap \gamma \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\gamma)$ holds
- Residual composition $\alpha \rhd \beta : X \rightharpoonup Z$ is a morphism s.t. $\forall \delta : X \rightharpoonup Z.\ \delta \sqsubseteq \alpha \rhd \beta$ iff $\alpha^\sharp\delta \sqsubseteq \beta$

Examples

The category $\boldsymbol{Rel(L)}$ of sets and $\boldsymbol{L}$-relations

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The category $Rel(L)$ of sets and L-relations

X, Y: sets, L: complete Heyting algebra

- *L-relation* $\alpha\colon X \rightharpoondown Y$ is a mapping $\alpha\colon X \times Y \to L$
- For $\alpha, \alpha'\colon X \rightharpoondown Y$ the ordering $\alpha \sqsubseteq \alpha'$ is defined by $\forall x \in X \forall y \in Y. \alpha(x, y) \leq \alpha'(x, y)$
- For $\alpha\colon X \rightharpoondown Y$, $\beta\colon Y \rightharpoondown Z$ the composition $\alpha\beta\colon X \rightharpoondown Z$ is defined by $\alpha\beta(x, z) = \vee_{y \in Y}(\alpha(x, y) \wedge \beta(y, z))$

Examples

The category $\boldsymbol{Rel}(\boldsymbol{L})$ of sets and $\boldsymbol{L}$-relations

The category $\boldsymbol{FRel}$ ($\cong \boldsymbol{Rel}([0, 1])$) of sets and fuzzy relations

The category $\boldsymbol{Rel}$ ($\cong \boldsymbol{Rel}(\{0, 1\})$) of sets and relations

Background

$\boldsymbol{Rel(L)}$, $\boldsymbol{FRel}$, $\boldsymbol{Rel}$: Objects are sets

Points of objects are available

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Only homsets have a rich structure

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notion of points and point axioms
for Dedekind categories have been studied

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notion of points and point axioms (PA), (PA_*)
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In addition to this, related axioms such as

- axioms of totality (Tot), (Tot_*)
- axiom of subobject (Sub)
- axiom of complement (Ba)
- relational axiom of choice (AC)

have been introduced.

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- Summarise interrelations of point axioms and some related axioms
- Remark some fundamental facts on L-relations
- Study under Winter's sufficient conditions (Winter, M.: Complements in distributive allegories. RelMiCS 2009)

Basic Notions

$\alpha : X \rightharpoonup Y$

α:*univalent* $\iff \alpha^\sharp \alpha \sqsubseteq \mathrm{id}_Y$

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α:*tfn* $\iff \alpha$:univalent and total

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I: object

$$I\text{:}unit \iff 0_{II} \neq \mathrm{id}_I = \nabla_{II} \wedge \forall X.(\nabla_{XI} \nabla_{IX} = \nabla_{XX})$$

(substitute for a singleton set)

I-point

X: object, I: unit

A tfn $x\colon I \longrightarrow X$ is called an I-*point* of X

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X: object, I: unit

A tfn $x \colon I \rightarrow X$ is called an I*-point* of X

<u>Notations</u>

$$x \mathrel{\dot{\in}} X \iff x \text{ is an } I\text{-point of } X$$

$\rho \colon I \rightharpoondown X$

$$x \mathrel{\dot{\in}} \rho \iff x \mathrel{\dot{\in}} X \wedge x \sqsubseteq \rho$$

Point Axioms

(PA) $\forall X. (\nabla_{IX} = \sqcup_{x \dot{\in} X} x)$

(universal relation ∇_{IX} is the sup of all I-points of X)

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(PA$_*$) $\forall \rho \colon I \rightharpoonup X. (\rho = \sqcup_{x \dot{\in} \rho} x)$

(relation $\rho \colon I \rightharpoonup X$ is the sup of all I-points in ρ)

On Point Axiom (PA)

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- $\mathrm{id}_X = \sqcup_{x \dot{\in} X} x^\sharp x$
- $\forall \alpha, \alpha' : X \rightharpoonup Y.$
  $[\forall x \dot{\in} X.(x\alpha = x\alpha') \to (\alpha = \alpha')]$

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- $\forall \alpha, \alpha' : X \rightharpoonup Y.$
  $[\forall x \dot{\in} X.(x\alpha = x\alpha') \to (\alpha = \alpha')]$
- $\forall \mu, \mu' : X \rightharpoonup I.$
  $[\forall x \dot{\in} X.(x\mu = x\mu') \to (\mu = \mu')]$

Axioms of Totality

(Tot)

$$\forall X. [(\nabla_{IX} \neq 0_{IX}) \rightarrow (\mathrm{id}_I = \nabla_{IX}\nabla_{XI})]$$

(nonzero universal relation ∇_{IX} is total)

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(Tot$_*$)

$$\forall \rho : I \rightharpoonup X.\,[(\rho \neq 0_{IX}) \rightarrow (\mathrm{id}_I = \rho\rho^\sharp)]$$

(nonzero relation $\rho : I \rightharpoonup X$ is total)

Point Axiom and Totality

(PA) $\forall X. (\nabla_{IX} = \sqcup_{x \in X}^{\bullet} x)$
(PA$_*$) $\forall \rho : I \rightharpoonup X. (\rho = \sqcup_{x \in \rho}^{\bullet} x)$
(Tot) $\forall X. [(\nabla_{IX} \neq 0_{IX}) \rightarrow (\mathrm{id}_I = \nabla_{IX} \nabla_{XI})]$
(Tot$_*$) $\forall \rho : I \rightharpoonup X. [(\rho \neq 0_{IX}) \rightarrow (\mathrm{id}_I = \rho \rho^\sharp)]$

$$
\begin{array}{ccc}
\text{(PA)} & \longrightarrow & \text{(Tot)} \\
\uparrow & & \uparrow \\
\text{(PA}_*\text{)} & \longrightarrow & \text{(Tot}_*\text{)}
\end{array}
$$

On Totality (Tot_*)

(Tot_*) $\forall \rho\colon I \rightharpoonup X. [(\rho \neq 0_{IX}) \to (\mathrm{id}_I = \rho\rho^\sharp)]$

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- All I-points $x\colon I \longrightarrow X$ are atoms of $\mathcal{D}(I, X)$

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- $\mathcal{D}(I, I) = \{0_{II}, \mathrm{id}_I\}$

On Totality (Tot_*)

(Tot_*) $\forall \rho\colon I \rightharpoonup X. [(\rho \neq 0_{IX}) \to (\mathrm{id}_I = \rho\rho^\sharp)]$

The following three are equivalent to (Tot_*)

- All I-points $x\colon I \to X$ are atoms of $\mathcal{D}(I, X)$
- $\mathcal{D}(I, I) = \{0_{II}, \mathrm{id}_I\}$
- All nonzero relations $\alpha\colon X \rightharpoonup Y$ satisfy $\nabla_{XX}\alpha\nabla_{YY} = \nabla_{XY}$ (Tarski rule)

(Tot$_*$) in $Rel(L)$

(Tot$_*$) $\forall \rho \colon I \rightharpoonup X. \, [(\rho \neq 0_{IX}) \rightarrow (\mathrm{id}_I = \rho\rho^\sharp)]$
L: complete Heyting algebra

If $Rel(L)$ satisfies (Tot$_*$), $L \cong \{0, 1\}$

(Tot$_*$) in $Rel(L)$

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If $Rel(L)$ satisfies (Tot$_*$), $L \cong \{0, 1\}$

Assuming (Tot$_*$), an L-relation becomes an ordinary relation

Representation Theorem

(PA) $\forall X. (\nabla_{IX} = \sqcup_{x \dot{\in} X} x)$

(Tot$_*$) $\forall \rho \colon I \rightharpoonup X. [(\rho \neq 0_{IX}) \to (\mathrm{id}_I = \rho\rho^\sharp)]$

For a Dedekind category $\mathcal{D}$ satisfying (PA) and (Tot$_*$) there exists a full and faithful functor which embeds $\mathcal{D}$ into $\boldsymbol{Rel}$.

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For a Dedekind category $\mathcal{D}$ satisfying (PA) and (Tot$_*$) there exists a full and faithful functor which embeds $\mathcal{D}$ into Rel.

The functor is determined by

$$\begin{array}{rcl} \chi \colon \mathcal{D} & \longrightarrow & \mathit{Rel}(\mathcal{D}(I,I)) \\ X & \mapsto & \{x \mid x \dot{\in} X\} \end{array}$$

$$\begin{array}{ccccc} X & & \chi(X) \times \chi(Y) & \ni & (x,y) \\ \downarrow \alpha & \mapsto & \downarrow \chi(\alpha) & & \updownarrow \\ Y & & \mathcal{D}(I,I) & \ni & x\alpha y^\sharp \end{array}$$

Axiom of complement

(Ba) $\forall \alpha : X \rightharpoondown Y.\, (\alpha \sqcup \neg\alpha = \nabla_{XY})$

(each relation has a complement)

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(Tot$_*$) $\forall \rho\colon I \rightharpoondown X.\, [(\rho \neq 0_{IX}) \to (\mathrm{id}_I = \rho\rho^\sharp)]$

<u>Theorem</u> (PA) $\wedge$ (Tot$_*$) $\longrightarrow$ (Ba)

Axiom of complement

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Theorem (PA) $\wedge$ (Tot$_*$) $\longrightarrow$ (Ba)

$\alpha' : X \rightharpoondown Y$ s.t. $\chi(\alpha') = \chi(\alpha)^-$ is a complement of $\alpha : X \rightharpoondown Y$ since Rel satisfies (Ba) and χ is full and faithful.

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$\alpha' : X \rightharpoondown Y$ s.t. $\chi(\alpha') = \chi(\alpha)^-$ is a complement of $\alpha : X \rightharpoondown Y$ since $\boldsymbol{Rel}$ satisfies (Ba) and χ is full and faithful.

(another proof is given in the paper)

Axiom of complement

$$\text{(Ba)}\ \forall\alpha : X \rightharpoonup Y.\,(\alpha \sqcup \neg\alpha = \nabla_{XY})$$

(each relation has a complement)

$\text{(PA)}\ \forall X.\,(\nabla_{IX} = \sqcup_{x \dot\in X}\, x)$

$(\text{Tot}_*)\ \forall\rho : I \rightharpoonup X.\,[(\rho \neq 0_{IX}) \to (\mathrm{id}_I = \rho\rho^\sharp)]$

Theorem (PA) $\wedge$ (Tot_*) $\longrightarrow$ (Ba)

$(\text{PA}_*)\ \forall\rho : I \rightharpoonup X.\,(\rho = \sqcup_{x \dot\in \rho}\, x)$

Remark Since $(\text{PA}_*) \longrightarrow \text{(PA)} \wedge (\text{Tot}_*)$,

$$(\text{PA}_*) \longrightarrow \text{(Ba)}$$

Axiom of Choice

(AC) $\forall \alpha : X \rightharpoonup Y.$
$$[(\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp) \longrightarrow \exists f : X \to Y. (f \sqsubseteq \alpha)]$$
(each total relation contains at least one tfn)

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(AC) fails in FRel.

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(AC) fails in FRel.

$(0,1)$: open real interval
$\rho \colon I \times (0,1) \longrightarrow [0,1]$ defined by $\rho(*, t) = t$
is total but no tfn contains.

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(AC) fails in FRel.

(AC) holds in $\mathit{Rel}(L)$ if L is a complete Boolean algebra.

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(AC) fails in $\boldsymbol{FRel}$.

(AC) holds in $\boldsymbol{Rel}(\boldsymbol{L})$ if $\boldsymbol{L}$ is a complete Boolean algebra.

Studying fuzzy relation, (AC) should not be assumed

Winter (2009)

- For all equivalence relations $\theta : X \rightharpoonup X$, there exists a tfn (*total splitting*) $s : X \to Q$ such that $s^\sharp s = \mathrm{id}_Q$ and $ss^\sharp = \theta$.
- For all pairs of objects X and Y there exists an object (*relational sum*) $X + Y$ together with a pair of tfns $i : X \to X + Y$ and $j : Y \to X + Y$ satisfying

$$i^\sharp i \sqcup j^\sharp j = \mathrm{id}_{X+Y}, \quad ii^\sharp = \mathrm{id}_X, \quad jj^\sharp = \mathrm{id}_Y, \quad \text{and} \quad ij^\sharp = 0_{XY}.$$

- For all pairs of objects X and Y there exists an object (*relational product*) $X \times Y$ together with a pair of tfns $p : X \times Y \to X$ and $q : X \times Y \to Y$ satisfying

$$pp^\sharp \sqcap qq^\sharp = \mathrm{id}_{X\times Y}, \quad p^\sharp p = \mathrm{id}_X, \quad q^\sharp q = \mathrm{id}_Y, \quad \text{and} \quad p^\sharp q = \nabla_{XY}.$$

Theorem (AC) $\longrightarrow$ (Ba) in $\mathcal{D}$ with total splittings, relational sums and relational products.

(Tot_*) and (PA_*)

(AC) $\forall \alpha : X \rightharpoonup Y. [(\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp) \to \exists f : X \to Y. (f \sqsubseteq \alpha)]$
(Ba) $\forall \alpha : X \rightharpoonup Y. (\alpha \sqcup \neg\alpha = \nabla_{XY})$
(Tot_*) $\forall \rho : I \rightharpoonup X. [(\rho \neq 0_{IX}) \to (\mathrm{id}_I = \rho\rho^\sharp)]$
(PA_*) $\forall \rho : I \rightharpoonup X. (\rho = \sqcup_{x \bullet\!\in \rho}\, x)$

Proposition

$(\mathrm{AC}) \wedge (\mathrm{Ba}) \wedge (\mathrm{Tot}_*) \leftrightarrow (\mathrm{AC}) \wedge (\mathrm{PA}_*)$

(Tot$_*$) and (PA$_*$)

(AC) $\forall \alpha\colon X \rightharpoonup Y. [(\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp) \to \exists f\colon X \to Y. (f \sqsubseteq \alpha)]$
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(Tot$_*$) $\forall \rho\colon I \rightharpoonup X. [(\rho \neq 0_{IX}) \to (\mathrm{id}_I = \rho\rho^\sharp)]$
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Proposition (New)

(AC) $\wedge$ (Ba) $\wedge$ (Tot$_*$) $\leftrightarrow$ ~~(AC) $\wedge$~~ (PA$_*$)

(Tot_*) and (PA_*)

(AC) $\forall \alpha : X \rightharpoondown Y. [(\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp) \to \exists f : X \to Y. (f \sqsubseteq \alpha)]$
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(Tot_*) $\forall \rho : I \rightharpoondown X. [(\rho \neq 0_{IX}) \to (\mathrm{id}_I = \rho\rho^\sharp)]$
(PA_*) $\forall \rho : I \rightharpoondown X. (\rho = \bigsqcup_{x \dot{\in} \rho} x)$

Proposition (New)

(AC) $\wedge$ (Ba) $\wedge$ (Tot_*) $\leftrightarrow$ ~~(AC) $\wedge$~~ (PA_*)

and Theorem in the last slide implies

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(Ba) $\forall \alpha : X \rightharpoondown Y . (\alpha \sqcup \neg\alpha = \nabla_{XY})$
(Tot$_*$) $\forall \rho : I \rightharpoondown X . [(\rho \neq 0_{IX}) \to (\mathrm{id}_I = \rho\rho^\sharp)]$
(PA$_*$) $\forall \rho : I \rightharpoondown X . (\rho = \sqcup_{x \dot\in \rho}\, x)$

Proposition (New)

(AC) ∧ (Ba) ∧ (Tot$_*$) ↔ ~~(AC) ∧~~ (PA$_*$)

and Theorem in the last slide implies

Corollary If $\mathcal{D}$ with total splittings, relational sums and relational products satisfies (AC),

$$(\mathrm{Tot}_*) \leftrightarrow (\mathrm{PA}_*)$$

(Tot_*) and (PA)

(AC) $\forall \alpha : X \rightharpoondown Y. [(\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp) \rightarrow \exists f : X \to Y. (f \sqsubseteq \alpha)]$
(Ba) $\forall \alpha : X \rightharpoondown Y. (\alpha \sqcup \neg\alpha = \nabla_{XY})$
(Tot_*) $\forall \rho : I \rightharpoondown X. [(\rho \neq 0_{IX}) \rightarrow (\mathrm{id}_I = \rho\rho^\sharp)]$
(PA) $\forall X. (\nabla_{IX} = \sqcup_{x \dot{\in} \nabla_{IX}} x)$

Remark In $\mathcal{D}$ with (AC) $\wedge$ (Ba)
$(\mathrm{Tot}_*) \longrightarrow$ (PA) holds but (PA) $\longrightarrow (\mathrm{Tot}_*)$ fails

(Tot$_*$) and (PA)

(AC) $\forall \alpha : X \rightharpoonup Y. [(\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp) \rightarrow \exists f : X \rightarrow Y. (f \sqsubseteq \alpha)]$
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(Tot$_*$) $\forall \rho : I \rightharpoonup X. [(\rho \neq 0_{IX}) \rightarrow (\mathrm{id}_I = \rho\rho^\sharp)]$
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Remark In $\mathcal{D}$ with (AC) $\wedge$ (Ba)
(Tot$_*$) $\longrightarrow$ (PA) holds but (PA) $\longrightarrow$ (Tot$_*$) fails

(Tot$_*$) $\longrightarrow$ (PA) follows from (PA$_*$) $\longrightarrow$ (PA) and (AC) $\wedge$ (Ba) $\wedge$ (Tot$_*$) $\leftrightarrow$ (PA$_*$)

(Tot_*) and (PA)

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(Tot$_*$) $\longrightarrow$ (PA) holds but (PA) $\longrightarrow$ (Tot$_*$) fails

$B = \{0, a, \neg a, 1\}$: Boolean algebra

(Tot$_*$) and (PA)

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(Tot$_*$) $\longrightarrow$ (PA) holds but (PA) $\longrightarrow$ (Tot$_*$) fails

$B = \{0, a, \neg a, 1\}$: Boolean algebra
$Rel(B)$ satisfies (AC) $\wedge$ (Ba) $\wedge$ (PA).

(Tot_*) and (PA)

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Remark In $\mathcal{D}$ with (AC) $\wedge$ (Ba)
$(\mathrm{Tot}_*) \longrightarrow$ (PA) holds but (PA) $\longrightarrow$ (Tot_*) fails

$B = \{0, a, \neg a, 1\}$: Boolean algebra
$Rel(B)$ satisfies (AC) $\wedge$ (Ba) $\wedge$ (PA).
Though $\rho : I \times I \longrightarrow B$ defined by
$\rho(*, *) = a$ is nonzero, $\rho\rho^\sharp = \rho \neq \mathrm{id}_I$.

(Tot_*) and (PA) New

(AC) $\forall\alpha\colon X \rightharpoondown Y.\,[(\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp) \rightarrow \exists f\colon X \rightarrow Y.\,(f \sqsubseteq \alpha)]$
(Tot_*) $\forall\rho\colon I \rightharpoondown X.\,[(\rho \neq 0_{IX}) \rightarrow (\mathrm{id}_I = \rho\rho^\sharp)]$
(PA) $\forall X.\,(\nabla_{IX} = \sqcup_{x \dot\in \nabla_{IX}} x)$

<u>Remark</u> In $\mathcal{D}$ with total splittings, relational sums, relational products and (AC), $(\mathsf{Tot}_*) \longrightarrow$ (PA) holds but (PA) $\longrightarrow (\mathsf{Tot}_*)$ fails

(Tot$_*$) and (PA) New

(AC) $\forall \alpha : X \rightharpoonup Y. [(\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp) \to \exists f : X \to Y. (f \sqsubseteq \alpha)]$
(Tot$_*$) $\forall \rho : I \rightharpoonup X. [(\rho \neq 0_{IX}) \to (\mathrm{id}_I = \rho\rho^\sharp)]$
(PA) $\forall X. (\nabla_{IX} = \sqcup_{x \dot{\in} \nabla_{IX}} x)$

<u>Remark</u> In $\mathcal{D}$ with total splittings, relational sums, relational products and (AC), (Tot$_*$) $\longrightarrow$ (PA) holds but (PA) $\longrightarrow$ (Tot$_*$) fails

$Rel(B)$: same as the last slide
$Eq(Rel(B))$: Dedekind category obtained by total split construction
(Proof of $Eq(Rel(B)) \models$(AC) has not been finished in time)

Summary of Interrelations

$$
\begin{array}{ccccccc}
(\mathrm{PA}) \wedge (\mathrm{Tot}_*) & & (\mathrm{PA}) & \rightarrow & (\mathrm{NE}) & \rightarrow & (\mathrm{Tot}) \\
\updownarrow & & \uparrow & & \uparrow & & \uparrow \\
(\mathrm{Ba}) \wedge (\mathrm{NE}_*) & \leftrightarrow & (\mathrm{PA}_*) & \rightarrow & (\mathrm{NE}_*) & \rightarrow & (\mathrm{Tot}_*) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
(\mathrm{Ba}) \wedge (\mathrm{NE}) \wedge (\mathrm{Sub}) & \leftrightarrow & (\mathrm{PA}) \wedge (\mathrm{Sub}) & \rightarrow & (\mathrm{NE}) \wedge (\mathrm{Sub}) & \rightarrow & (\mathrm{Tot}) \wedge (\mathrm{Sub})
\end{array}
$$

$(\mathrm{AC}) \wedge (\mathrm{Ba}) \wedge (\mathrm{Tot}_*) \leftrightarrow$ ~~$(\mathrm{AC}) \wedge$~~ (PA_*)

Details of (NE), (NE_*) and (Sub) are omitted in this talk.