Continuous Relations and Richardson's Theorem

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RAMiCS 13 @ Cambridge
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Background (Richardson's theorem)

Richardson, D.: Tassellations with local transformations. J. Computer and System Sciences 6, 373–388 (1972)

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- characterisation of transition relations determined by local rule of CA

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- characterisation of transition relations determined by local rule of CA
- using a notions of "topology" and "continuous relations" on the set of configurations.

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Problems (Richardson's theorem)

"Topological" notions are introduced indirectly (via points of accumulations)

Richardson, D.: Tassellations with local transformations. J. Computer and System Sciences 6, 373–388 (1972)

References for Solution

Continuity of relations has been studied in

Brattka, V. and Hertling, P.: Continuity and computability of relations. Informatik Berichte vol. 164. FernUniversität in Hagen (1994)

Ziegler, M.: Relative computability and uniform continuity of relations. Logic seminar 2011, Technische Universität Darmstadt (2011).

What we have done

Reformulation of

- continuity of relations due to Brattke-Hertling
- nondeterministic CAs

by morphisms of $\boldsymbol{Rel}$ (relations)

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Proof of Richardson's theorem using topological and relational devices

Notations

X, Y: sets

$$Rel(X, Y) = \{\alpha \mid \alpha \subseteq X \times Y\}$$

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$\lfloor \alpha \rfloor = \alpha\alpha^\sharp \sqcap \mathrm{id}_X$ (domain of $\alpha : X \rightharpoondown Y$)

Basic Notions

$\alpha : X \rightharpoondown Y$

α:*univalent* $\iff \alpha^\sharp \alpha \sqsubseteq \mathrm{id}_Y$

$(\ \forall x \in X \forall y, y' \in Y. (x, y) \in \alpha \wedge (x, y') \in \alpha \implies y = y'\ )$

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α:*tfn* $\iff \alpha$:univalent and total

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α:*rectangular* $\iff \alpha \nabla_{YX} \alpha \sqsubseteq \alpha$
$(\exists W \subseteq X \exists Z \subseteq Y. \alpha = W \times Z)$

Elements

X: set, I: singleton set

Elements of X is identified with tfn from I to X

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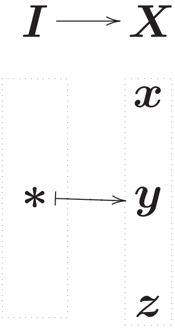
Elements of X is identified with tfn from I to X

$$y \in \{x, y, z\} \iff$$



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Subsets of X is identified with relations in $Rel(I, X)$

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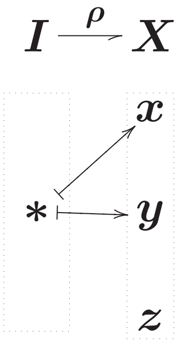

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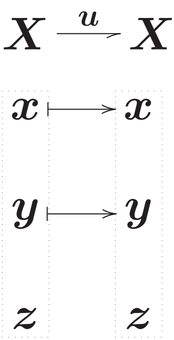

Topological Space

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$$(X, \mathcal{O}(X))$$

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- finite intersection

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Note: $0_{IX}, \nabla_{IX} \in \mathcal{O}(X)$

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inverse image of a open set is the intersection of α's domain and some open set

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$$\iff \nabla_{IX}\alpha \in \mathcal{O}(Y) \wedge \nabla_{IY}\alpha^\sharp \in \mathcal{O}(X)$$

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basis element of the usual product topology of $X \times Y$

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basis elements of the usual product topology of $X \times Y$

$\alpha : X \rightharpoonup Y$ is *closed* $\iff \alpha^-$ is open

Nondeterministic CA over Group

Y^X: the set of mappings from a set X to a set Y

Q: nonempty finite set, G: group

- $q \in Q$: *state*
- $c \in Q^G$: *configuration*
- $\delta : Q^G \rightharpoondown Q^G$: *transition relation*

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$\{0,1\}^{F(\{g\})} \ni$

$\cdots$	g^{-3}	g^{-2}	g^{-1}	g^0	g^1	g^2	g^3	$\cdots$
$\cdots$	1	0	1	0	1	0	1	$\cdots$

$\downharpoonleft \delta$ $\qquad \downarrow$

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$\cdots$	g^{-3}	g^{-2}	g^{-1}	g^0	g^1	g^2	g^3	$\cdots$
$\cdots$	$*$	1	$*$	1	$*$	1	$*$	$\cdots$

($* = 0$ or 1)

Projection

$U \subseteq G$, $x \in G$

$p_U : Q^G \to Q^U$ denotes the projection

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		$\cdots$	1	0	1	0	1	0	1	$\cdots$

$\downarrow p_{\{g^{-1},g^{1}\}}$ $\qquad\qquad \mapsto$

$\{0,1\}^{\{g^{-1},g^{1}\}} \ni$

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1	1

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$$\{0,1\}^{F(\{g\})} \ni \begin{array}{c|c|c|c|c|c|c|c|c} \cdots & g^{-3} & g^{-2} & g^{-1} & g^{0} & g^{1} & g^{2} & g^{3} & \cdots \\ \hline \cdots & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \end{array}$$

$$\downarrow p_{\{g^{-1},g^{1}\}} \qquad \mapsto$$

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$\cdots$	1	0	1	0	1	0	1	$\cdots$

$\downarrow p_{g^{-1}}$ $\quad \mapsto$

$\{0,1\} \ni$

g^{-1}
1

Shift Function

$x \in G$

shift function $t_x \colon Q^G \to Q^G$ is defined by

$$\forall a \in G.\, t_x p_a = p_{x^{-1}a}$$

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Local Rule

N: finite subset of G

$\lambda\colon Q^N \rightharpoondown Q$ is called a *local rule*

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$\{0,1\}^{\{g^{-1},g^1\}}$	g^{-1} g^1	g^{-1} g^1	g^{-1} g^1	g^{-1} g^1
	0 0	0 1	1 0	1 1
$\downharpoonleft \boldsymbol{\lambda}$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\{0,1\}$	$*$		0	1

($* = 0$ or 1)

Transition Determined by Local Rule

$\lambda\colon Q^N \rightharpoonup Q$: local rule

$\tau_\lambda\colon Q^G \rightharpoonup Q^G$ defined by

$$\tau_\lambda = \sqcap_{x\in G} t_{x^{-1}} p_N \lambda p_x^\sharp$$

is called the *transition relation determined by* λ

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$\{0,1\}^{\{g^{-1},g^{1}\}}$	g^{-1} g^{1}	g^{-1} g^{1}	g^{-1} g^{1}	g^{-1} g^{1}	
	0 0	0 1	1 0	1 1	
$\downarrow \lambda$	$\mapsto$	$\mapsto$	$\mapsto$	$\mapsto$	
$\{0,1\}$	$*$		0	1	($*=0$ or 1)

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Transition Determined by Local Rule

$\{0,1\}^{\{g^{-1},g^{1}\}} \xrightarrow{\lambda} \{0,1\}$

g^{-1}	g^{1}	λ
0	0	$*$
0	1	
1	0	0
1	1	1

($* = 0$ or 1)

$$\tau_\lambda = \sqcap_{x\in G} t_{x^{-1}} p_N \lambda p^\sharp_x$$

$\{0,1\}^{F(\{g\})}$

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$\cdots$	1	0	1	0	1	0	1	$\cdots$

$\downarrow t_{g^2}$

$\{0,1\}^{F(\{g\})}$

$\cdots$	-1	g^{0}	g^{1}	g^{2}	g^{3}	4	5	$\cdots$
$\cdots$	1	0	1	0	1	0	1	$\cdots$

$\downarrow p_{\{g^{-1},g^{1}\}}$

$\{0,1\}^{\{g^{-1},g^{1}\}}$

g^{-1}	g^{1}
1	1

$\downarrow \lambda$

$\{0,1\}$

1

$\downarrow p^\sharp_{g^{-2}}$

$\{0,1\}^{F(\{g\})}$

$\cdots$	g^{-3}	g^{-2}	g^{-1}	g^{0}	g^{1}	g^{2}	g^{3}	$\cdots$
$\cdots$	$*$	1	$*$	$*$	$*$	$*$	$*$	$\cdots$

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0	0		0	1		1	0		1	1
$\downarrow$ $*$			$\downarrow$			$\downarrow$ 0			$\downarrow$ 1	

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$\downarrow \tau_\lambda$

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- $\mathcal{O}(Q) = Rel(I, Q)$
  $((\cong \wp(Q))$ discrete topology)
- $\mathcal{O}(Q^G)$ is the least topology s.t. for each $x \in G\ p_x \colon Q^G \to Q$ is continuous,
  i.e. $\forall \rho \in \mathcal{O}(Q).\ \rho p_x^\sharp \in \mathcal{O}(Q^G)$
  (product topology)

Richardson's Theorem (1/2)

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$\tau_\lambda \colon Q^G \to Q^G$ satisfies

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  (commutative with shifts)

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- $\sqcap_{x \in G} \tau_\lambda p_x p_x^\sharp = \tau_\lambda$

  (determines states of each cell independently)

Richardson's Theorem (1/2)

$\lambda : Q^N \rightharpoondown Q$: local rule

$\tau_\lambda : Q^G \rightharpoondown Q^G$ satisfies

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  (determines states of each cell independently)

- closed and continuous

Richardson's Theorem (2/2)

For transition $\delta\colon Q^G \rightharpoonup Q^G$ s.t.

- $\forall x \in G.\, t_x\delta = \delta t_x$
  (commutative with shifts)
- $\sqcap_{x\in G}\delta p_x p_x^\sharp = \delta$
  (determines states of each cell independently)
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  (determines states of each cell independently)
- closed and continuous

there exists a local rule $\lambda\colon Q^N \rightharpoonup Q$ s.t.

$$\delta = \lfloor\delta\rfloor\tau_\lambda.$$

(δ: restriction of τ_λ)

Another Result by Richardson

If local rule $\lambda \colon Q^N \rightharpoonup Q$ is a tfn, then

$$\forall \text{ finite } S \subseteq G.\, \tau_S \text{ is surjective}$$
$$\implies \tau_\lambda \text{ is surjective}$$

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where $\tau_S\colon Q^{SN} \rightharpoonup Q^S$ is defined by

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and where $SN = \{xy \mid x \in S,\, y \in N\}$

Another Result by Richardson

If local rule $\lambda\colon Q^N \rightharpoonup Q$ is a tfn, then

$$\forall \text{ finite } S \subseteq G.\, \tau_S \text{ is surjective}$$
$$\implies \tau_\lambda \text{ is surjective}$$

where $\tau_S\colon Q^{SN} \rightharpoonup Q^S$ is defined by

$$\tau_S = \sqcap_{x \in G} p^\sharp_{SN} t_{x^{-1}} p_N \lambda p^\sharp_x p_S$$

and where $SN = \{xy \mid x \in S,\, y \in N\}$

(τ_S: restriction of τ_λ)

Slight Extension

If local rule $\lambda\colon Q^N \rightharpoonup Q$ is total, then

$$\forall \text{ finite } S \subseteq G.\, \tau_S \text{ is surjective}$$
$$\implies \tau_\lambda \text{ is surjective}$$

Conclusion

What we have done:

- Investigation of the continuity of relations between topological spaces using relational notation
- Development fundamental properties of continuous relations
- Proof of Richardson's theorem using relational and topological devices
- Extension of a tiny part of Richardson's results

Conclusion

What we are going to do:

- More investigation of nondeterministic CA

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Relational and topological devices developed in this research should be useful for it!