Some Relational Style Laws of Linear Algebra

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Abstract. We present a few laws of linear algebra inspired by laws of relation algebra. The linear algebra laws are obtained from the relational ones by replacing union, intersection, composition and converse by the linear algebra operators of addition, Hadamard product, composition and transposition. Many of the modified expressions hold directly or with minor alterations.

1 Introduction

The purpose of this short note is to present a few laws of linear algebra that are similar to corresponding laws of relation algebra. The starting point is the remark that matrices with 0, 1 entries are relations. Let Q and R be such matrices. Then their Hadamard product $Q \cdot R$, i.e., their entrywise arithmetic multiplication, is their intersection. The standard addition $Q+R$ and composition (multiplication) QR are not quite the union and relational composition, but they are not so far from that. Transpose R^{T} and conjugate transpose $R^{\dagger}$ are exactly the converse of R. Our goal is to study what happens when the relational operators of a relational law are replaced by the linear algebra operators, and what happens when arbitrary matrices are used instead of relations.

Our purpose is to augment the repertoire of point-free laws of linear algebra, an endeavour in the spirit of the work of Macedo and Oliveira [4, 5]. Some, if not most, of these laws are already known, but we nevertheless feel the "relational twist" is worth exploring.

Section 2 presents the notation and some basic laws. Section 3 introduces domain-like operators. Sections 4 and 5 are about direct sums and direct products. We conclude in Section 6. We assume knowledge of the relational material that is used below, which can be found in [7, 8]. There are numerous textbooks on linear algebra; see, e.g., [6].

2 Basic Laws

We consider finite matrices over the complex numbers. In the sequel, the term *relations* refers to matrices with 0, 1 entries. Variables A, B, C denote arbitrary matrices and P, Q, R denote relations. Matrix composition is denoted by juxtaposition, as is standard in linear algebra. The other operators are matrix addition $+$, Hadamard product $\cdot$ (entrywise multiplication: $(A \cdot B)_{i,j} = A_{i,j} \times B_{i,j}$), conjugate transpose †, transpose $^\mathsf{T}$, identity matrix $\mathbb{I}$ and zero matrix $\mathbf{0}$ ($\mathbf{0}_{i,j} = 0$ for all i, j). For relations, they are union $\cup$, intersection $\cap$, composition $;$, converse $^\smile$ and universal relation $\mathbb{T}$ ($\mathbb{T}_{i,j} = 1$ for all i, j). The size of a matrix with m rows and n columns is indicated by $m \leftrightarrow n$, occasionally as a subscript. The unary operators have precedence over the binary ones. The order of increasing precedence for the binary operators is $(+, \cup)$, $(\cdot, \cap)$, (composition, $;$).

A matrix A is a *relation* iff $A \cdot A = A$. For a relation R, $R^\smile = R^\mathsf{T} = R^\dagger$. The universal relation $\mathbb{T}$ is the neutral element of the Hadamard product: $A \cdot \mathbb{T} = A$.

Using matrix composition on relations rather than relational composition gives a more "quantitative" result. Indeed, $(QR)_{i,j}$ is the number of paths from i to j by following Q and then R, rather than simply indicating whether there is a path or not. In particular, all entries of the matrix $\mathbb{T}_{l\leftrightarrow m}\mathbb{T}_{m\leftrightarrow n}$ are m, the size of the intermediate set (rows for the first matrix, columns for the second).

A matrix A is *diagonal* iff $A \cdot \mathbb{I} = A$. A relation R is *univalent* iff $R^\dagger R$ is diagonal; the entry $(R^\dagger R)_{j,j}$ is the number of rows i such that iRj, which gives a measure of the degree of non-injectivity. Relations together with the Hadamard product can be used to impose "shapes" to arbitrary matrices. For instance, if R is univalent, then $A \cdot R$ is a matrix with at most one non-zero entry in each row; thus, A has at most one non-zero entry in each row iff $A = A \cdot R$ for some univalent relation R. Instead of univalent relations, one may use equivalence relations, difunctional relations, symmetric relations, etc. to impose shapes.

Let us say that matrix A is *unitarget* iff $A = A \cdot R$ for some univalent relation R. If A is unitarget , then

$$(A \cdot A)(B \cdot C) = AB \cdot AC\ . \tag{1}$$

Thus, if R is a univalent relation, one has from (1) and $R \cdot R = R$ that $R(B \cdot C) = RB \cdot RC$, a well-known law of linear algebra that generalises the relational law that univalent relations left distribute over intersection: $R;(P \cap Q) = R;P \cap R;Q$.

We prove (1). Assume A is unitarget.

$$\begin{aligned}
& ((A \cdot A)(B \cdot C))_{i,j} \\
= \;& (\textstyle\sum k \mid (A \cdot A)_{i,k} \times (B \cdot C)_{k,j}) \\
= \;& (\textstyle\sum k \mid A_{i,k} \times A_{i,k} \times B_{k,j} \times C_{k,j}) \\
= \;& \quad \langle \text{ If } A_{i,k} = 0 \text{ for all } k\text{, choose an arbitrary } k_i\text{; otherwise, let } k_i \\
& \quad\ \ \text{be the unique } k \text{ such that } A_{i,k} \neq 0 \ \rangle \\
& A_{i,k_i} \times A_{i,k_i} \times B_{k_i,j} \times C_{k_i,j} \\
= \;& (A_{i,k_i} \times B_{k_i,j}) \times (A_{i,k_i} \times C_{k_i,j}) \\
= \;& (\textstyle\sum k \mid A_{i,k} \times B_{k,j}) \times (\textstyle\sum k \mid A_{i,k} \times C_{k,j}) \\
= \;& (AB)_{i,j} \times (AC)_{i,j} \\
= \;& (AB \cdot AC)_{i,j} \qquad \Box
\end{aligned}$$

A diagonal matrix whose diagonal entries are all equal codes for a scalar. We thus say that a matrix D is a *scalar* iff $D = D \cdot \mathbb{I}$ and $D\mathbb{T} = \mathbb{T}D$. Since $D\mathbb{T}$

and $\mathbb{T}D$ are then matrices whose entries are all equal, they could also be used to code for a scalar.

Various simple laws follow. If the linear operators are replaced by the corresponding relational ones (as described in the introduction), the relational laws that inspired these laws are easily recognised.

Proposition 1. *1.* $(A \cdot \mathbb{I})(A \cdot \mathbb{I}) = (A \cdot \mathbb{I}) \cdot (A \cdot \mathbb{I}) = A \cdot A \cdot \mathbb{I}$*, i.e., for diagonal matrices, composition and Hadamard product coincide.*

2. $A \cdot \mathbb{I} = A^{\mathsf{T}} \cdot \mathbb{I}$*, with special case* $A\mathbb{T} \cdot \mathbb{I} = \mathbb{T}A^{\mathsf{T}} \cdot \mathbb{I}$.

3. $(A\mathbb{T} \cdot B)C = A\mathbb{T} \cdot BC$*, with special case* $(A\mathbb{T} \cdot \mathbb{I})C = A\mathbb{T} \cdot C$.

4. $A(B\mathbb{T} \cdot C) = (A \cdot \mathbb{T}B^{\mathsf{T}})C$.

5. If D is diagonal, then $D = D\mathbb{T} \cdot \mathbb{I} = \mathbb{T}D \cdot \mathbb{I}$.

6. If D is a scalar, then $DA = AD$ for all A.

7. $\mathbb{T}A\mathbb{T} \cdot \mathbb{I}$ *is a scalar. Note that* $\mathbb{T}A\mathbb{T}$ *is a matrix whose entries are all equal to the sum of the entries of A. Thus,* $\mathbb{T}A\mathbb{T} \cdot \mathbb{I}$ *is a scalar matrix denoting the sum of the elements of A.*

8. $(A \cdot B)\mathbb{T} = (AB^{\mathsf{T}} \cdot \mathbb{I})\mathbb{T}$*. A consequence of this law is* $\mathbb{T}(A \cdot B)\mathbb{T} = \mathbb{T}(AB^{\mathsf{T}} \cdot \mathbb{I})\mathbb{T}$*, which says that the sum of the entries of $A \cdot B$ is the trace of AB^{T}.*

9. If R is a univalent relation, then $R^{\dagger}(RA \cdot B) = A \cdot R^{\dagger}B$.

10. Let R be a relation. If either $\mathbf{0} \leq A$*,* $\mathbf{0} \leq B$ *or* $A \leq \mathbf{0}$*,* $B \leq \mathbf{0}$*, then* $RA \cdot B \leq R(A \cdot R^{\dagger}B)$*. If either* $\mathbf{0} \leq A$*,* $B \leq \mathbf{0}$ *or* $A \leq \mathbf{0}$*,* $\mathbf{0} \leq B$*, then* $R(A \cdot R^{\dagger}B) \leq RA \cdot B$*. This is similar to the Dedekind rule for relations:* $R;P \cap Q \subseteq R;(P \cap R^{\smile};Q)$.

Proof. 1. This is direct from the definitions.

2. This is direct from the definitions.

3.
$$
\begin{aligned}
& ((A\mathbb{T} \cdot B)C)_{i,j} \\
= \;& (\textstyle\sum k \mid (A\mathbb{T})_{i,k} \times B_{i,k} \times C_{k,j}) \\
= \;& \qquad \langle\ (A\mathbb{T})_{i,k} = (A\mathbb{T})_{i,j} \text{ for all } i,j \ \ \& \ \text{Distributivity}\ \rangle \\
& (A\mathbb{T})_{i,j} \times (\textstyle\sum k \mid B_{i,k} \times C_{k,j}) \\
= \;& (A\mathbb{T})_{i,j} \times (BC)_{i,j} \\
= \;& (A\mathbb{T} \cdot BC)_{i,j}
\end{aligned}
$$

4.
$$
\begin{aligned}
& A(B\mathbb{T} \cdot C) \\
= \;& \qquad \langle\ \text{Item 3 of this proposition}\ \rangle \\
& A(B\mathbb{T} \cdot \mathbb{I})C \\
= \;& \qquad \langle\ \text{Item 2 of this proposition}\ \rangle \\
& A(\mathbb{T}B^{\mathsf{T}} \cdot \mathbb{I})C \\
= \;& \qquad \langle\ \text{Dual of item 3 of this proposition}\ \rangle \\
& (A \cdot \mathbb{T}B^{\mathsf{T}})C
\end{aligned}
$$

5. Since D is diagonal, $D = D \cdot \mathbb{I}$. Thus, $D_{i,j} = 0 = (D\mathbb{T} \cdot \mathbb{I})_{i,j}$ if $i \neq j$. If $i = j$, then $(D\mathbb{T} \cdot \mathbb{I})_{i,i} = (D\mathbb{T})_{i,i} = (\sum k \mid D_{i,k}) = (\sum k \mid (D \cdot \mathbb{I})_{i,k}) = D_{i,i}$. The proof of $D = \mathbb{T}D \cdot \mathbb{I}$ is similar.

6. Using item 5 of this proposition, the fact that $D\mathbb{T} = \mathbb{T}D$ because D is a scalar, and item 3 of this proposition and its dual, we have

$$DA = (D\mathbb{T} \cdot \mathbb{I})A = D\mathbb{T} \cdot A = \mathbb{T}D \cdot A = A(\mathbb{T}D \cdot \mathbb{I}) = AD \ .$$

7. Firstly, $(\mathbb{T}A\mathbb{T} \cdot \mathbb{I}) \cdot \mathbb{I} = \mathbb{T}A\mathbb{T} \cdot \mathbb{I}$. Secondly, $(\mathbb{T}A\mathbb{T} \cdot \mathbb{I})\mathbb{T} = \mathbb{T}A\mathbb{T} \cdot \mathbb{T} = \mathbb{T}(\mathbb{T}A\mathbb{T} \cdot \mathbb{I})$ by item 3 of this proposition and its dual.

8. $$\begin{aligned} & ((A \cdot B)\mathbb{T})_{i,j} \\ = \ & (\sum k \mid (A \cdot B)_{i,k}) \\ = \ & (\sum k \mid A_{i,k} \times B_{i,k}) \\ = \ & (\sum k \mid A_{i,k} \times (B^{\mathsf{T}})_{k,i}) \\ = \ & (AB^{\mathsf{T}})_{i,i} \\ = \ & (\sum k \mid (AB^{\mathsf{T}} \cdot \mathbb{I})_{i,k}) \\ = \ & (AB^{\mathsf{T}} \cdot \mathbb{I})\mathbb{T} \end{aligned}$$

9. $$\begin{aligned} & (R^{\dagger}(RA \cdot B))_{i,j} \\ = \ & (\sum k \mid (R^{\dagger})_{i,k} \times (RA \cdot B)_{k,j}) \\ = \ & (\sum k, l \mid (R^{\dagger})_{i,k} \times R_{k,l} \times A_{l,j} \times B_{k,j}) \\ = \ & \quad \langle \text{ Because } R \text{ is univalent, } (R^{\dagger})_{i,k} \times R_{k,l} = R_{k,i} \times R_{k,l} = 0 \text{ if } i \neq l. \text{ Otherwise, } (R^{\dagger})_{i,k} \times R_{k,l} = (R^{\dagger})_{i,k} \times R_{k,i} = (R^{\dagger})_{i,k}, \text{ because } R \text{ is a relation } \rangle \\ & (\sum k \mid (R^{\dagger})_{i,k} \times A_{i,j} \times B_{k,j}) \\ = \ & (A \cdot R^{\dagger}B)_{i,j} \end{aligned}$$

10. Assume either $\mathbf{0} \leq A$, $\mathbf{0} \leq B$ or $A \leq \mathbf{0}$, $B \leq \mathbf{0}$.

$$\begin{aligned} & (R(A \cdot R^{\dagger}B))_{i,j} \\ = \ & (\sum k \mid R_{i,k} \times (A \cdot R^{\dagger}B)_{k,j}) \\ = \ & (\sum k, l \mid R_{i,k} \times A_{k,j} \times (R^{\dagger})_{k,l} \times B_{l,j}) \\ \geq \ & \quad \langle \text{ By the assumption, } A_{k,j} \times B_{l,j} \geq 0 \ \ \& \text{ Because } R \text{ is a relation, } R_{i,k} \times (R^{\dagger})_{k,i} = R_{i,k} \geq 0 \ \rangle \\ & (\sum k \mid R_{i,k} \times A_{k,j} \times B_{i,j}) \\ = \ & (RA \cdot B)_{i,j} \end{aligned}$$

When either $\mathbf{0} \leq A$, $B \leq \mathbf{0}$ or $A \leq \mathbf{0}$, $\mathbf{0} \leq B$, the proof is similar, except that this assumption reverses the inequality. □

3 Domain-like Operators

Like in relation algebra, the information content of a vector can be obtained as a diagonal matrix. If vector V has type $n \leftrightarrow 1$, then the diagonal matrix $V\mathbb{T}_{1\leftrightarrow n} \cdot \mathbb{I}$ corresponds to V (its diagonal contains the same elements as V, in the same order). Given a diagonal matrix $D_{n\leftrightarrow n}$, the corresponding vector is $D\mathbb{T}_{n\leftrightarrow 1}$. A vector V of type $n \leftrightarrow 1$ is a *unit vector* iff $V^{\dagger}V = 1$ $(= \mathbb{T}_{1\leftrightarrow 1})$.

Using the above correspondence between vectors and diagonal matrices, we say that a diagonal matrix D is a *unit diagonal matrix* iff $\mathbb{T}D^{\dagger}D\mathbb{T} = \mathbb{T}$ (which is equivalent to $\mathbb{T}(D^{\dagger} \cdot D)\mathbb{T} = \mathbb{T}$).

A common operation in linear algebra is the multiplication of a matrix A by a vector V, giving the vector AV as a result. The dual operation $V^{\dagger}A$ is also frequent. In order to carry the same operations at the level of diagonal matrices, we introduce two operators, the *column-sum operator* ${}^{\Sigma}A$ and *row-sum operator* A^{Σ}, defined by

$$ {}^{\Sigma}A = A\mathbb{T} \cdot \mathbb{I} \ , \qquad A^{\Sigma} = \mathbb{T}A \cdot \mathbb{I} \ . \tag{2} $$

A simple example explains how the operators work and where their names come from:

$$ {}^{\Sigma}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & 0 \\ 0 & c+d \end{bmatrix} \ , \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{\Sigma} = \begin{bmatrix} a+c & 0 \\ 0 & b+d \end{bmatrix} \ . $$

Notice the similarity of these definitions with the relation algebraic definitions of the *domain* operator $\ulcorner R = R\,;\mathbb{T}\cap\mathbb{I}$ and *codomain* operator $R\urcorner = \mathbb{T}\,;R\cap\mathbb{I}$, which encode the usual domain and codomain of a relation R as subidentity relations. Such domain and codomain operators have been investigated thoroughly in the more abstract setting of semirings and Kleene algebra [1, 2]. It turns out that they share some properties with the column-sum and row-sum operators. There are some differences, though, as the following table shows.

	Linear algebra	Relation algebra
(a)		$\ulcorner R;R = R$
(b)		$\ulcorner R;\ulcorner R = \ulcorner R$
(c)	${}^{\Sigma}(AB) = {}^{\Sigma}(A({}^{\Sigma}B))$	$\ulcorner(Q\,;R) = \ulcorner(Q\,;\ulcorner R)$
(d)	${}^{\Sigma}A({}^{\Sigma}A) = {}^{\Sigma}A \cdot {}^{\Sigma}A$	$\ulcorner Q\,;\ulcorner R = \ulcorner Q \cap \ulcorner R$
(e)	${}^{\Sigma}({}^{\Sigma}AB) = {}^{\Sigma}A({}^{\Sigma}B)$	$\ulcorner(\ulcorner Q\,;R) = \ulcorner Q\,;\ulcorner R$
(f)	${}^{\Sigma}(A+B) = {}^{\Sigma}A + {}^{\Sigma}B$	$\ulcorner(Q \cup R) = \ulcorner Q \cup \ulcorner R$
(g)	${}^{\Sigma\Sigma}A = {}^{\Sigma}A$	$\ulcorner\ulcorner R = \ulcorner R$
(h)	${}^{\Sigma}(A^{\dagger}) = (A^{\Sigma})^{\dagger}$	$\ulcorner(R^{\smile}) = R\urcorner$
(i)	${}^{\Sigma}(A^{\mathsf{T}}) = A^{\Sigma}$	$\ulcorner(R^{\smile}) = R\urcorner$
(j)	$A\mathbb{T} \cdot B = {}^{\Sigma}AB$	$Q\,;\mathbb{T} \cap R = \ulcorner Q\,;R$

(3)

We prove the less obvious laws.

1. *Proof of (3c).* By (2), Proposition 1(3) and neutrality of $\mathbb{T}$ for the Hadamard product,

$$ {}^{\Sigma}(A({}^{\Sigma}B)) = A(B\mathbb{T} \cdot \mathbb{I})\mathbb{T} \cdot \mathbb{I} = A(B\mathbb{T} \cdot \mathbb{T}) \cdot \mathbb{I} = AB\mathbb{T} \cdot \mathbb{I} = {}^{\Sigma}(AB) \ . $$

2. *Proof of (3e).* By (2), Proposition 1(3) and definition of the Hadamard product,

$$^{\Sigma}(^{\Sigma}AB) = (A\mathbb{T}\cdot\mathbb{I})B\mathbb{T}\cdot\mathbb{I} = A\mathbb{T}\cdot B\mathbb{T}\cdot\mathbb{I} = (A\mathbb{T}\cdot\mathbb{I})\cdot(B\mathbb{T}\cdot\mathbb{I}) = {}^{\Sigma}A(^{\Sigma}B) \ .$$

$\square$

Unlike for relation algebra laws (3a) and (3b), $^{\Sigma}AA = A$ and $^{\Sigma}A(^{\Sigma}A) = {}^{\Sigma}A$ do not hold, since, e.g.,

$$^{\Sigma}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad {}^{\Sigma}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\left({}^{\Sigma}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \ .$$

If t is a relational test (a subidentity), then forward and backward diamond modal operators can be defined by $|R\rangle t = {}^{\ulcorner}(R\,;t)$ and $\langle R|t = (t\,;R)^{\urcorner}$ [3]. The corresponding linear algebra expressions are $^{\Sigma}(AD)$ and $(DA)^{\Sigma}$, where D is a diagonal matrix. Both $^{\Sigma}(AD)$ and $(DA)^{\Sigma}$ are diagonal matrices. If one views D as a description of the "content" or "amplitude" of a state, then $(DA)^{\Sigma}$, for instance, is the content or amplitude of the state obtained from state D by transformation A. Using (2), Proposition 1(3) or its transposition dual, and neutrality of $\mathbb{T}$ for the Hadamard product, we get $^{\Sigma}(AD)\mathbb{T} = AD\mathbb{T}$ and $\mathbb{T}(DA)^{\Sigma} = \mathbb{T}DA$. This show that the operation $^{\Sigma}(AD)$ involving diagonal matrices corresponds to the expression AV, involving vectors, and similarly for $(DA)^{\Sigma}$ and $V^{\mathsf{T}}A$; in fact, this is just the same as for the domain and codomain operators of relation algebra.

A matrix A is unitary iff $A^{\dagger}A = AA^{\dagger} = \mathbb{I}$. If A is unitary and V is a unit vector, then AV is a unit vector. The corresponding property for diagonal matrices is that $^{\Sigma}(AD)$ is a unit diagonal matrix if A is unitary and D is a unit diagonal matrix. This is proved as follows.

$$
\begin{array}{ll}
 & \mathbb{T}(^{\Sigma}(AD))^{\dagger}(^{\Sigma}(AD))\mathbb{T} \\
= & \qquad \langle \text{ (2) } \rangle \\
 & \mathbb{T}(AD\mathbb{T}\cdot\mathbb{I})^{\dagger}(AD\mathbb{T}\cdot\mathbb{I})\mathbb{T} \\
= & \qquad \langle \text{ Linear algebra } \rangle \\
 & \mathbb{T}(\mathbb{T}D^{\dagger}A^{\dagger}\cdot\mathbb{I})(AD\mathbb{T}\cdot\mathbb{I})\mathbb{T} \\
= & \qquad \langle \text{ Proposition 1(3) and its transposition dual } \rangle \\
 & (\mathbb{T}D^{\dagger}A^{\dagger}\cdot\mathbb{T})(AD\mathbb{T}\cdot\mathbb{T}) \\
= & \qquad \langle \text{ Neutrality of } \mathbb{T} \text{ for the Hadamard product } \rangle \\
 & \mathbb{T}D^{\dagger}A^{\dagger}AD\mathbb{T} \\
= & \qquad \langle\ A \text{ is unitary } \rangle \\
 & \mathbb{T}D^{\dagger}D\mathbb{T} \\
= & \qquad \langle\ D \text{ is a unit diagonal matrix } \rangle \\
 & \mathbb{T}
\end{array}
$$

$\square$

4 Direct Sums

Relational *direct sums* are axiomatised as a pair (σ_1, σ_2) of injections satisfying the following axioms:

$$\text{(a) } \sigma_1 ; \sigma_1^{\smile} = \mathbb{I} \ , \ \text{(b) } \sigma_2 ; \sigma_2^{\smile} = \mathbb{I} \ , \ \text{(c) } \sigma_1 ; \sigma_2^{\smile} = \mathbf{0} \ , \ \text{(d) } \sigma_1^{\smile} ; \sigma_1 \cup \sigma_2^{\smile} ; \sigma_2 = \mathbb{I} \ . \quad (4)$$

Because σ_1, σ_2 are injective functions and because $\sigma_1^{\smile} ; \sigma_1$ and $\sigma_2^{\smile} ; \sigma_2$ are disjoint, the relational operators can be replaced by the linear ones, allowing other solutions in addition to the relational ones:

$$\text{(a) } \sigma_1 \sigma_1^{\dagger} = \mathbb{I} \ , \ \text{(b) } \sigma_2 \sigma_2^{\dagger} = \mathbb{I} \ , \ \text{(c) } \sigma_1 \sigma_2^{\dagger} = \mathbf{0} \ , \ \text{(d) } \sigma_1^{\dagger} \sigma_1 + \sigma_2^{\dagger} \sigma_2 = \mathbb{I} \ . \quad (5)$$

As for relations, these direct sums allow one to build matrices by blocks (i.e., by combining smaller matrices). We refer to [4, 5] for an extensive study of this construct.

Equations (4) define σ_1 and σ_2 up to isomorphism only. Other solutions can be obtained by suitable permutations of the rows and columns of the relations σ_1 and σ_2. With Equations (5), even more solutions are possible. If A, A_1 and A_2 are unitary, then $(A_1^{\dagger} \sigma_1 A, A_2^{\dagger} \sigma_2 A)$ is also a direct sum satisfying (5). This amounts to having a direct sum in a different orthonormal basis.

5 Direct Products

Relational *direct products* are axiomatised as a pair (π_1, π_2) of projections satisfying the following equations:

$$\text{(a) } \pi_1^{\smile} ; \pi_1 = \mathbb{I} \ , \ \text{(b) } \pi_2^{\smile} ; \pi_2 = \mathbb{I} \ , \ \text{(c) } \pi_1^{\smile} ; \pi_2 = \mathbb{T} \ , \ \text{(d) } \pi_1 ; \pi_1^{\smile} \cap \pi_2 ; \pi_2^{\smile} = \mathbb{I} \ . \quad (6)$$

These equations define π_1 and π_2 up to isomorphism. For example, the following relations π_1 of type $3 \times 2 \leftrightarrow 3$ and π_2 of type $3 \times 2 \leftrightarrow 2$ provide a solution:

$$\pi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad \pi_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad .$$

If we use this solution in the linear algebra variant of (6), we see that $\pi_1^{\dagger} \pi_2 = \mathbb{T}$ and $\pi_1 \pi_1^{\dagger} \cdot \pi_2 \pi_2^{\dagger} = \mathbb{I}$ hold, but not $\pi_1^{\dagger} \pi_1 = \mathbb{I}$ and $\pi_2^{\dagger} \pi_2 = \mathbb{I}$, since

$$\pi_1^{\dagger} \pi_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \pi_2^{\dagger} \pi_2 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad .$$

But $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbb{T}_{3\leftrightarrow 2}\mathbb{T}_{2\leftrightarrow 3} \cdot \mathbb{I}$ and $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{T}_{2\leftrightarrow 3}\mathbb{T}_{3\leftrightarrow 2} \cdot \mathbb{I}$, which leads to the appropriate laws for defining direct products with the linear algebra operators, where π_1 has type $m \times n \leftrightarrow m$ and π_2 type $m \times n \leftrightarrow n$:

$$\begin{array}{ll} \text{(a) } \pi_1^\dagger \pi_1 = \mathbb{T}_{m\leftrightarrow n}\mathbb{T}_{n\leftrightarrow m} \cdot \mathbb{I} \ , & \text{(c) } \pi_1^\dagger \pi_2 = \mathbb{T} \ , \\ \text{(b) } \pi_2^\dagger \pi_2 = \mathbb{T}_{n\leftrightarrow m}\mathbb{T}_{m\leftrightarrow n} \cdot \mathbb{I} \ , & \text{(d) } \pi_1 \pi_1^\dagger \cdot \pi_2 \pi_2^\dagger = \mathbb{I} \ . \end{array} \tag{7}$$

Then $\pi_1^\dagger \pi_1$ and $\pi_2^\dagger \pi_2$ are diagonal matrices whose entries in the diagonal are n and m, respectively.

In relation algebra, vectorisation of a relation R is obtained by $\mathsf{vec}(R) = (\pi_1 ; R \cap \pi_2) ; \mathbb{T}$. If π_1 and π_2 are relations, this works for arbitrary matrices A and the linear algebra operators. For instance, with $A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$,

$$\mathsf{vec}(A) = (\pi_1 A \cdot \pi_2)\mathbb{T}_{2\leftrightarrow 1} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} .$$

Unvectorisation works as well:

$$A = \pi_1^\dagger (\mathsf{vec}(A)\mathbb{T}_{1\leftrightarrow n} \cdot \pi_2) \ . \tag{8}$$

Thus,

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \left(\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} .$$

Compared to what happens with relations, there is the additional constraint that the $\mathbb{T}$ used for vectorisation must have one column and that used for unvectorisation must have one row.

The proof of (8) follows.

$$\begin{array}{ll} & \pi_1^\dagger (\mathsf{vec}(A)\mathbb{T}_{1\leftrightarrow n} \cdot \pi_2) \\ = & \pi_1^\dagger ((\pi_1 A \cdot \pi_2)\mathbb{T}_{n\leftrightarrow 1}\mathbb{T}_{1\leftrightarrow n} \cdot \pi_2) \end{array}$$

$$
\begin{array}{ll}
= & \pi_1^\dagger((\pi_1 A \cdot \pi_2)\mathbb{T}_{n\leftrightarrow n} \cdot \pi_2) \\
= & \qquad \langle \text{ Proposition 1(3,8) } \ \& \ \text{ Because } \pi_2 \text{ is a relation, } \pi_2^{\mathsf{T}} = \pi_2^\dagger \ \rangle \\
 & \pi_1^\dagger((\pi_1 A \pi_2^\dagger \cdot \mathbb{I})\mathbb{T}_{n\leftrightarrow n} \cdot \mathbb{I})\pi_2 \\
= & \qquad \langle \ \pi_1 A \pi_2^\dagger \cdot \mathbb{I} \text{ is diagonal } \ \& \ \text{ Proposition 1(5) } \rangle \\
 & \pi_1^\dagger(\pi_1 A \pi_2^\dagger \cdot \mathbb{I})\pi_2 \\
= & \qquad \langle \ \pi_1 \text{ and } \pi_2 \text{ are univalent } \ \& \ \text{ Proposition 1(9) and its dual } \rangle \\
 & A \cdot \pi_1^\dagger \pi_2 \\
= & \qquad \langle \text{ (7c) } \ \& \ \mathbb{T} \text{ is neutral for the Hadamard product } \rangle \\
 & A \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square
\end{array}
$$

Given size-compatible projections π_1 and π_2, the *Kronecker product* $A \otimes B$ can now be defined:

$$
A \otimes B = \pi_1 A \pi_1^\dagger \cdot \pi_2 B \pi_2^\dagger \ . \tag{9}
$$

This is the standard Kronecker product of linear algebra. For instance, with the π_1 and π_2 given at the beginning of this section,

$$
\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \otimes \begin{bmatrix} j & k \\ l & m \end{bmatrix} = \begin{bmatrix} a\times j & a\times k & b\times j & b\times k & c\times j & c\times k \\ a\times l & a\times m & b\times l & b\times m & c\times l & c\times m \\ d\times j & d\times k & e\times j & e\times k & f\times j & f\times k \\ d\times l & d\times m & e\times l & e\times m & f\times l & f\times m \\ g\times j & g\times k & h\times j & h\times k & i\times j & i\times k \\ g\times l & g\times m & h\times l & h\times m & i\times l & i\times m \end{bmatrix} \ .
$$

The following laws and their dual under † are satisfied when π_1 and π_2 are relations:

$$
\begin{array}{ll}
\text{(a)} & (A\pi_1^\dagger \cdot B\pi_2^\dagger)\pi_1 = A \cdot B\mathbb{T} \ , \\
\text{(b)} & (A\pi_1^\dagger \cdot B\pi_2^\dagger)\pi_2 = A\mathbb{T} \cdot B \ , \\
\text{(c)} & (A\pi_1^\dagger \cdot B\pi_2^\dagger)(\pi_1 C \cdot \pi_2 D) = AC \cdot BD \ , \\
\text{(d)} & (A \otimes B)(C \otimes D) = (AC) \otimes (BD) \ , \\
\text{(e)} & \text{If } A \text{ and } B \text{ are invertible, so is } A \otimes B \ , \\
\text{(f)} & \text{If } A \text{ and } B \text{ are unitary, so is } A \otimes B \ .
\end{array} \tag{10}
$$

6 Conclusion

We plan to continue the exploration of similar laws inspired by those or relation and Kleene algebra. In addition, we need to identify a small set of basic formulae and derive the others from them in a pointfree way; for our taste, too many of the proofs of the above properties were done using indexes. Finally, we intend to look at applications in the areas of quantum automata and program derivation.

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