Relational Concepts in Social Choice

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Kenneth Arrow's impossibility theorem

When voters have ≥ 3 alternatives, no rank order voting system can convert their preferences into a community-wide ranking when one postulates these "obvious" criteria:

- unrestricted domain: . . . not just in special cases
- non-dictatorship: No single voter possesses the power to always determine the group's preference.
- Pareto efficiency: If every voter prefers alternative X over alternative Y, then so does the group.
- independence of irrelevant alternatives: If every voter's preference between X and Y remains unchanged, then so for the group's preference (even if preferences between other pairs like X and Z, Y and Z, or Z and W change).

Criticizing the concept of an order

- initially just linear orders
- highjump competition leads to preorder/weakorder
- strictorder fits better in the hierarchy than orders
- orders with threshold
- moving to preferences
- proceeding further to choice mappings

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Matrix orientation

§ 7. Darstellung linearer Räume durch lineare Gleichungen.

Den im vorigen Paragraphen für lineare Gleichungssysteme erhaltenen Ergebnissen, insbesondere dem Satz 5, kann man noch eine andere Deutung geben. Wir fassen zu dem Zweck n Zahlen, die einem vorgegebenen Gleichungssystem der Gestalt

$$(1) \qquad \begin{array}{c} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1 , \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2 , \\ \cdots\cdots\cdots\cdots\cdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m \end{array}$$

$$(1) \qquad \sum_{k=1}^{n} a_{ik} x_k = b_i , \qquad i = 1, 2, \ldots, n .$$

Wir wollen fürs erste voraussetzen, daß die einfache Matrix

$$(2) \qquad \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

Matrix orientation

Stoer/Bulirsch: Numerische Mathematik

Beispiel: Die Jordan-Matrix

$$
J = \left[\begin{array}{ccc|cc|cc|c|c}
1 & 1 & & & & & & & \\
 & 1 & 1 & & & & 0 & & \\
 & & 1 & & & & & & \\
\hline
 & & & 1 & 1 & & & & \\
 & & & & 1 & & & & \\
\hline
 & & & & & -1 & 1 & & \\
 & & & & & & -1 & & \\
\hline
 & & 0 & & & & & -1 & \\
\hline
 & & & & & & & & -1
\end{array}\right]
$$

Matrix orientation

Stoer/Bulirsch: Numerische Mathematik

6.5.4 Reduktion auf Hessenberg-Gestalt

Es wurde bereits in Abschnitt 6.5.1 bemerkt, daß man eine gegebene $n \times n$-Matrix A mittels $n-2$ Householdermatrizen T_i ähnlich auf Hessenberggestalt B transformieren kann

$$A := A_0 \to A_1 \to \dots \to A_{n-2} = B, \qquad A_i = T_i^{-1} A_{i-1} T_i.$$

Wir wollen nun einen zweiten Algorithmus dieser Art beschreiben, bei dem als Transformationsmatrizen T_i Permutationsmatrizen

$$P_{rs} = \begin{bmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & & & 1 & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & & \\ & & & 1 & & & 0 & & \\ & & & & & & & 1 & \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow r \\ \\ \\ \\ \leftarrow s \\ \\ \\ \\ \end{matrix}$$

und Eliminationsmatrizen der Form

$$G_j = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & l_{j+1,j} & 1 & & \\ & & \vdots & & \ddots & \\ & & l_{nj} & & & 1 \end{bmatrix}, \qquad |l_{ij}| \le 1,$$

benutzt werden. Diese Matrizen haben die Eigenschaft

$$P_{rs}^{-1} = P_{rs}$$

(6.5.4.1)
$$G_j^{-1} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & -l_{j+1,j} & 1 & & \\ & & \vdots & & \ddots & \\ & & -l_{nj} & & & 1 \end{bmatrix}.$$

Eine Linksmultiplikation $P_{rs}^{-1}A$ von A mit $P_{rs}^{-1} = P_{rs}$ bewirkt eine Vertauschung der Zeilen r und s von A, eine Rechtsmultiplikation AP_{rs} eine Vertauschung der Spalten r und s von A. Eine Linksmultiplikation $G_j^{-1}A$ von A mit G_j^{-1} bewirkt, daß für $r = j+1, j+2, \dots, n$ das l_{rj}-fache der Zeile j von Zeile r der Matrix A abgezogen wird, während eine Rechtsmultiplikation AG_j bedeutet, daß für $r = j+1, \dots, n$ das l_{rj}-fache der Spalte r zur Spalte j von A addiert wird.

Um A schrittweise mittels Ähnlichkeitstransformationen des betrachteten Typs auf Hessenberggestalt zu transformieren, gehen wir so vor: Wir setzen $A = A_0$ und nehmen induktiv an, daß A_{i-1} bereits eine Matrix ist, deren erste $i-1$ Spalten Hessenberggestalt haben:

(6.5.4.2)
$$A_{i-1} = \left[\begin{array}{cccc|c|ccc} * & \cdots & \cdots & * & * & * & \cdots & * \\ * & \ddots & & \vdots & \vdots & \vdots & & \vdots \\ & \ddots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & & * & * & * & * & \cdots & * \\ \hline 0 & \cdots & 0 & * & * & * & \cdots & * \\ \hline & & & & * & \vdots & & \vdots \\ & & & & \vdots & \vdots & & \vdots \\ 0 & & & & * & * & \cdots & * \end{array}\right]$$
$\underbrace{\qquad\qquad}_{i-1}$

Membership ε

$$U = \varepsilon;e \qquad e = \mathtt{syq}(\varepsilon, U)$$

	{}	{a}	{b}	{a,b}	{c}	{a,c}	{b,c}	{a,b,c}		U
a	0	1	0	1	0	1	0	1		1
b	0	0	1	1	0	0	1	1		0
c	0	0	0	0	1	1	1	1		1

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} = e^{\mathsf{T}}$$

	{}	{a}	{b}	{a,b}	{c}	{a,c}	{b,c}	{a,b,c}		e
{}	1	1	1	1	1	1	1	1		0
{a}	0	1	0	1	0	1	0	1		0
{b}	0	0	1	1	0	0	1	1		0
{a,b}	0	0	0	1	0	0	0	1		0
{c}	0	0	0	0	1	1	1	1		0
{a,c}	0	0	0	0	0	1	0	1		1
{b,c}	0	0	0	0	0	0	1	1		0
{a,b,c}	0	0	0	0	0	0	0	1		0

ε $\qquad$ U $\qquad$ $\Omega = \overline{\varepsilon^{\mathsf{T}};\overline{\varepsilon}}$ $\qquad$ e

Subset U and corresponding element e in the powerset via ε, Ω

$$\mathtt{syq}(A, B) := \overline{A^{\mathsf{T}};\overline{B}} \cap \overline{\overline{A}^{\mathsf{T}};B}$$

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Hierarchy of Order Concepts

	linear strictorder	weakorder	semiorder	interval-order	strictorder
transitive	$\bullet \mid \circ$	$\bullet$	$\bullet$	$\bullet$	$\bullet \mid \circ$
asymmetric	$\bullet \mid$	$\bullet \mid$	$\bullet \quad \circ$	$\bullet \quad \circ$	$\bullet \mid$
irreflexive	$\bullet \quad \circ$	$\bullet$	$\bullet \mid$	$\bullet \mid$	$\bullet \quad \circ$
Ferrers	$\bullet$	$\bullet$	$\bullet \mid \circ$	$\bullet \mid \circ$	—
semi-transitive	$\bullet$	$\bullet$	$\bullet \mid \circ$	—	—
negatively transitive	$\bullet$	$\bullet \mid$	—	—	—
semi-connex	$\bullet \mid \circ$	—	—	—	—

$$\text{But: } E \text{ Ferrers} \implies \overline{\mathbb{I}} \cap E \text{ Ferrers}$$
$$C \text{ Ferrers} \;\not\Longrightarrow\; \mathbb{I} \cup C \text{ Ferrers}$$

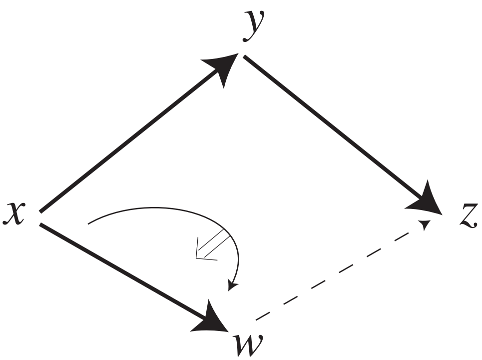
Graph drawing vs. matrix rearrangement

A relation R is called a **semiorder** if it is asymmetric $(R^\mathsf{T} \subseteq \overline{R})$ $\big[$or irreflexive $(R \subseteq \overline{\mathbb{I}})\big]$, together with the Ferrers $(R;\overline{R}^\mathsf{T};R \subseteq R)$ and the semi-transitivity property $(R;R;\overline{R}^\mathsf{T} \subseteq R)$. It is known that semiorders represent something like orders with threshold. Can we realize this?

	a	b	c	d	e	f	g	h	i	j	k	l	m
a	0	1	1	1	0	0	1	1	0	1	1	1	0
b	0	0	1	1	0	0	0	0	0	1	1	0	0
c	0	0	0	0	0	0	0	0	0	1	1	0	0
d	0	0	0	0	0	0	0	0	0	1	1	0	0
e	0	0	1	1	0	0	1	0	0	1	1	0	0
f	0	1	1	1	0	0	1	1	0	1	1	1	0
g	0	0	0	0	0	0	0	0	0	1	1	0	0
h	0	0	0	0	0	0	0	0	0	1	1	0	0
i	1	1	1	1	1	1	1	1	0	1	1	1	0
j	0	0	0	0	0	0	0	0	0	0	0	0	0
k	0	0	0	0	0	0	0	0	0	0	0	0	0
l	0	0	1	1	0	0	0	0	0	1	1	0	0
m	1	1	1	1	1	1	1	1	0	1	1	1	0

	i	m	a	f	e	b	l	h	g	c	d	j	k
i	0	0	1	1	1	1	1	1	1	1	1	1	1
m	0	0	1	1	1	1	1	1	1	1	1	1	1
a	0	0	0	0	0	1	1	1	1	1	1	1	1
f	0	0	0	0	0	1	1	1	1	1	1	1	1
e	0	0	0	0	0	0	0	0	1	1	1	1	1
b	0	0	0	0	0	0	0	0	0	1	1	1	1
l	0	0	0	0	0	0	0	0	0	1	1	1	1
h	0	0	0	0	0	0	0	0	0	0	0	1	1
g	0	0	0	0	0	0	0	0	0	0	0	1	1
c	0	0	0	0	0	0	0	0	0	0	0	1	1
d	0	0	0	0	0	0	0	0	0	0	0	1	1
j	0	0	0	0	0	0	0	0	0	0	0	0	0
k	0	0	0	0	0	0	0	0	0	0	0	0	0

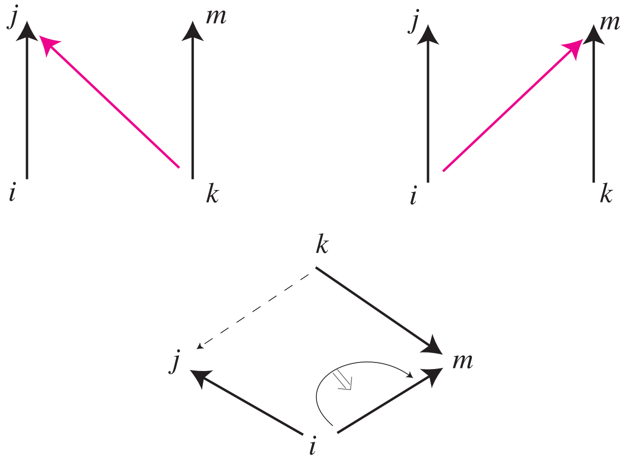
Graph interpretation of semitransitivity



$R{;}R{;}\overline{R}^{\mathsf{T}} \subseteq R$

This is an obviously homogeneous concept!

Graph interpretation of being Ferrers



This is a heterogeneous concept: $R;\overline{R}^{\mathsf{T}};R \subseteq R$

Graph drawing vs. matrix rearrangement

A relation R is called an **intervalorder** if it is asymmetric ($R^{\mathsf{T}} \subseteq \overline{R}$) $\left[\text{or irreflexive } (R \subseteq \overline{\mathbb{I}})\right]$, together with the Ferrers ($R;\overline{R}^{\mathsf{T}};R \subseteq R$) property.

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0	0	1	0	0	0	0	1	0	0	0	0	0
2	1	0	1	1	0	0	1	1	0	0	1	0	1
3	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	1	0	0	0	0	1	0	0	0	0	0
5	0	0	1	0	0	0	0	1	0	0	0	0	1
6	1	0	1	1	0	0	1	1	0	0	1	0	1
7	1	0	1	0	0	0	0	1	0	0	0	0	1
8	0	0	0	0	0	0	0	0	0	0	0	0	0
9	1	1	1	1	0	1	1	1	0	0	1	1	1
10	1	0	1	1	0	0	1	1	0	0	1	0	1
11	0	0	1	0	0	0	0	1	0	0	0	0	0
12	0	0	1	0	0	0	0	1	0	0	0	0	1
13	0	0	0	0	0	0	0	0	0	0	0	0	0

	5	9	10	2	6	12	4	7	11	1	13	3	8
9	0	0	0	1	1	1	1	1	1	1	1	1	1
2	0	0	0	0	0	0	1	1	1	1	1	1	1
6	0	0	0	0	0	0	1	1	1	1	1	1	1
10	0	0	0	0	0	0	1	1	1	1	1	1	1
7	0	0	0	0	0	0	0	0	0	1	1	1	1
5	0	0	0	0	0	0	0	0	0	0	1	1	1
12	0	0	0	0	0	0	0	0	0	0	1	1	1
1	0	0	0	0	0	0	0	0	0	0	0	1	1
4	0	0	0	0	0	0	0	0	0	0	0	1	1
11	0	0	0	0	0	0	0	0	0	0	0	1	1
3	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0

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From orders to preference structures

From any relation R — usually from a reflexive one called weak preference — one may obtain a preference structure with strict preference P, its converse P^T, indifference I, and incomparability J.

$$\alpha : R \mapsto (P, I, J)$$

$$\alpha(R) := (R \cap \overline{R}^\mathsf{T}, R \cap R^\mathsf{T}, \overline{R} \cap \overline{R}^\mathsf{T})$$

and back again

$$\beta : (P, I, J) \mapsto R$$

$$\beta(P, I, J) := P \cup I$$

From orders to preference structures

$$E = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & 1 & 1 \\ 4 & 0 & 0 & 0 & 1 \end{array}$$

ordering

$$R = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 1 & 1 & 0 & 1 & 1 \\ b & 0 & 1 & 0 & 0 & 0 \\ c & 1 & 0 & 1 & 1 & 0 \\ d & 0 & 0 & 0 & 1 & 1 \\ e & 0 & 1 & 0 & 1 & 1 \end{array}$$

weak preference

$$\begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 1 & 0 & 1 & 1 \\ b & 0 & 0 & 0 & 0 & 0 \\ c & 1 & 0 & 0 & 1 & 0 \\ d & 0 & 0 & 0 & 0 & 0 \\ e & 0 & 1 & 0 & 0 & 0 \end{array}$$

$P = R \cap \overline{R}^{\top}$

strict preference

$$\begin{array}{ccccc} a & b & c & d & e \\ \hline 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}$$

$P^{\top}$

converse

$$\begin{array}{ccccc} a & b & c & d & e \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}$$

$I = R \cap R^{\top}$

indifference

$$\begin{array}{ccccc} a & b & c & d & e \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}$$

$J = \overline{R}^{\top} \cap \overline{R}$

incomparability

Preference structures

Definition. Weak preference R and its asymmetric part P.

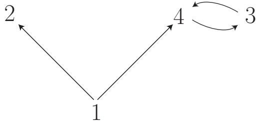
i) R **quasi-transitive** $:\iff$ P transitive

ii) R **acyclic** $:\iff$ $P^+ \subseteq \overline{P}^{\mathsf{T}}$

iii) R **acyclic**$_{\text{Sen}}$ $:\iff$ $P^+ \subseteq R$

iv) R **consistent** $:\iff$ $P;R^* \subseteq \overline{R}^{\mathsf{T}}$

v) P **progressively finite** $:\iff$ $\varepsilon \subseteq \mathbb{T};(\varepsilon \cap \overline{P;\varepsilon})$ $\square$



	1	2	3	4
1	0	1	0	1
2	0	0	0	0
3	0	0	0	1
4	0	0	1	0

P

	{}	{1}	{2}	{1,2}	{3}	{1,3}	{2,3}	{1,2,3}	{4}	{1,4}	{2,4}	{1,2,4}	{3,4}	{1,3,4}	{2,3,4}	{1,2,3,4}
1	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0
2	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
3	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0

$\varepsilon \cap \overline{P;\varepsilon}$

Illustrating the condition of being progressively finite; P is not

Preference structures

Proposition.

i)	R transitive	$\Longrightarrow$	R quasi-transitive
ii)	R transitive	$\Longrightarrow$	R consistent
iii)	R consistent	$\Longrightarrow$	R acyclic
iv)	R quasi-transitive	$\Longrightarrow$	R acyclic
v)	R acyclic$_{\text{Sen}}$	$\Longrightarrow$	R acyclic
vi)	R acyclic$_{\text{Sen}}$	$\not\Longleftarrow$	R acyclic

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Order-theoretic functionals

Let C resp. "$<$" be a strictorder.
An element is *a maximal element* in a subset U when exist no strictly greater element in this subset.

$$m \in U \;\wedge\; \overline{\exists e : m < e \wedge e \in U}.$$

$$\texttt{max}_C(U) := U \cap \overline{C;U}$$

Let E resp. "$\leq$" be an order.
The element $g \in U$ is called the **greatest element** of U if

$$g \in U \;\wedge\; \forall e : e \in U \rightarrow e \leq g.$$

$$\texttt{gre}_E(U) := U \cap \texttt{ubd}_E(U)$$

The element b is an **upper bound** of U if

$$\forall e : e \in U \rightarrow e \leq b.$$

$$\texttt{ubd}_E(U) := \overline{\overline{E}^\mathsf{T};U}$$

Order-theoretic functionals

Proposition. Let R be an arbitrary homogeneous relation.

i) $\mathtt{gre}_R(\varepsilon) \subseteq \mathtt{max}_R(\varepsilon)$

ii) R finite preorder $\implies \varepsilon \subseteq \mathbb{T};\mathtt{max}_R(\varepsilon)$

iii) R preorder $\implies \mathtt{gre}_R(\varepsilon) = \mathtt{max}_R(\varepsilon) \cap \mathbb{T};\mathtt{gre}_R(\varepsilon)$

Proposition. For an arbitrary homogeneous relation R

i) $\mathtt{syq}(\mathtt{gre}_R(\varepsilon), \varepsilon) \subseteq \Omega^\mathsf{T}$

ii) $\mathtt{gre}_R(\varepsilon);\Omega^\mathsf{T} \cap \varepsilon = \mathtt{gre}_R(\varepsilon)$

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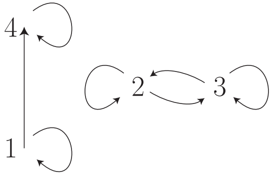
From preference structures to choice mappings

R



$R =$

	1	2	3	4
1	1	0	0	1
2	0	1	1	0
3	0	1	1	0
4	0	0	0	1

weak preference

$C =$

	{}	{1}	{2}	{1,2}	{3}	{1,3}	{2,3}	{1,2,3}	{4}	{1,4}	{2,4}	{1,2,4}	{3,4}	{1,3,4}	{2,3,4}	{1,2,3,4}
{}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{1}	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{2}	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
{1,2}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{3}	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
{1,3}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{2,3}	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
{1,2,3}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{4}	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
{1,4}	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
{2,4}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{1,2,4}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{3,4}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{1,3,4}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{2,3,4}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{1,2,3,4}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

choice mapping

Rational choice and revealed preference

X set of **conceivable states** with membership $\varepsilon : X \longrightarrow \mathbf{2}^X$
and powerset ordering $\Omega : \mathbf{2}^X \longrightarrow \mathbf{2}^X$

Definition

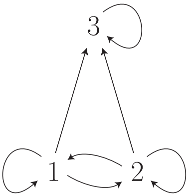
A relation $C : \mathbf{2}^X \longrightarrow \mathbf{2}^X$ that is univalent, total, and contracting, i.e., a function which satisfies $C \subseteq \Omega^{\mathsf{T}}$, is called a (generalized) **choice mapping**.

Better known are non-total concepts:

- **Sen choice function** if $C \subseteq \mathbb{T};\varepsilon, \quad C;\mathbb{T} = \varepsilon^{\mathsf{T}};\mathbb{T}$,
- **Suzumura choice function** if $C \subseteq \mathbb{T};\varepsilon, \quad C;\mathbb{T} \subseteq \varepsilon^{\mathsf{T}};\mathbb{T}$.

When non-total: $(X, \mathcal{S})$ is the **choice space**, with $\mathcal{S} := C;\mathbb{T}$

Choice function and choice mapping



	1	2	3
1	1	1	1
2	1	1	1
3	0	0	1

R

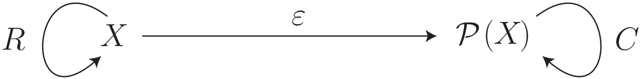
	{}	{1}	{2}	{1,2}	{3}	{1,3}	{2,3}	{1,2,3}
{}	1	0	0	0	0	0	0	0
{1}	0	1	0	0	0	0	0	0
{2}	0	0	1	0	0	0	0	0
{1,2}	0	0	0	1	0	0	0	0
{3}	0	0	0	0	1	0	0	0
{1,3}	0	0	0	0	1	0	0	0
{2,3}	0	0	0	0	1	0	0	0
{1,2,3}	0	0	0	0	1	0	0	0

choice mapping

	{}	{1}	{2}	{1,2}	{3}	{1,3}	{2,3}	{1,2,3}
{}	0	0	0	0	0	0	0	0
{1}	0	1	0	0	0	0	0	0
{2}	0	0	1	0	0	0	0	0
{1,2}	0	0	0	1	0	0	0	0
{3}	0	0	0	0	1	0	0	0
{1,3}	0	0	0	0	1	0	0	0
{2,3}	0	0	0	0	1	0	0	0
{1,2,3}	0	0	0	0	1	0	0	0

Sen choice function

Generating a choice mapping out of a preference



Definition

Given any homogeneous relation R, not necessarily an order or a preorder, we speak of its **corresponding**

i) **choice mapping** if

$$C = \mathtt{syq}(\mathtt{gre}_R(\varepsilon), \varepsilon),$$

ii) **Suzumura choice function** if

$$C = \mathtt{syq}(\mathtt{gre}_R(\varepsilon), \varepsilon) \cap \mathbb{T};\varepsilon,$$

iii) **Sen choice function** if

$$C = \mathtt{syq}(\mathtt{gre}_R(\varepsilon), \varepsilon) \cap \mathbb{T};\varepsilon \quad \text{and} \quad C;\mathbb{T} = \varepsilon^\mathsf{T};\mathbb{T}.$$

Order-theoretic functionals

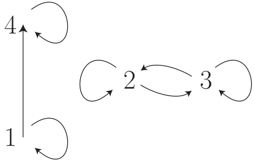
Proposition. Let be given a homogeneous relation R and its corresponding choice mapping $C = \mathtt{syq}(\mathtt{gre}_R(\varepsilon), \varepsilon)$. Then

i) C is indeed a mapping, i.e., total and univalent,
ii) $C \subseteq \Omega^\mathsf{T}$,
iii) $\varepsilon ; C^\mathsf{T} = \mathtt{gre}_R(\varepsilon)$,
iv) $\varepsilon ; C^\mathsf{T} = \varepsilon ; C^\mathsf{T} ; \Omega^\mathsf{T} \cap \varepsilon$.

Generating a choice mapping out of a preference

	1	2	3	4
1	1	0	0	1
2	0	1	1	0
3	0	1	1	0
4	0	0	0	1

R



	{}	{1}	{2}	{1,2}	{3}	{1,3}	{2,3}	{1,2,3}	{4}	{1,4}	{2,4}	{1,2,4}	{3,4}	{1,3,4}	{2,3,4}	{1,2,3,4}
1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0
3	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0

1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
2	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
3	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
4	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

$\mathtt{gre}_R(\varepsilon) := \varepsilon \cap \overline{\overline{R}^{\mathsf{T}} ; \varepsilon}$ $\qquad \varepsilon$

$C = \mathtt{syq}(\mathtt{gre}_R(\varepsilon), \varepsilon)$

	{}	{1}	{2}	{1,2}	{3}	{1,3}	{2,3}	{1,2,3}	{4}	{1,4}	{2,4}
{}	1	0	0	0	0	0	0	0	0	0	0
{1}	0	1	0	0	0	0	0	0	0	0	0
{2}	0	0	1	0	0	0	0	0	0	0	0
{1,2}	1	0	0	0	0	0	0	0	0	0	0
{3}	0	0	0	0	1	0	0	0	0	0	0
{1,3}	1	0	0	0	0	0	0	0	0	0	0
{2,3}	0	0	0	0	0	0	1	0	0	0	0
{1,2,3}	1	0	0	0	0	0	0	0	0	0	0
{4}	0	0	0	0	0	0	0	0	1	0	0
{1,4}	0	0	0	0	0	0	0	0	1	0	0
{2,4}	1	0	0	0	0	0	0	0	0	0	0
{1,2,4}	1	0	0	0	0	0	0	0	0	0	0
{3,4}	1	0	0	0	0	0	0	0	0	0	0
{1,3,4}	1	0	0	0	0	0	0	0	0	0	0
{2,3,4}	1	0	0	0	0	0	0	0	0	0	0
{1,2,3,4}	1	0	0	0	0	0	0	0	0	0	0

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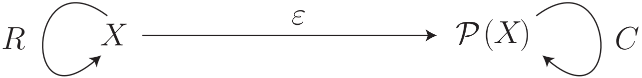
From choice to revealed preference

- Given any choice function $C : \mathbf{2}^X \longrightarrow \mathbf{2}^X$, the relation $R : X \longrightarrow X$ is said to **rationalize** C if

  $$\varepsilon ; C^{\mathsf{T}} = \mathtt{gre}_R(\varepsilon).$$

- If such an R exists, we call C a **rational** choice function.
- If this R is in addition an ordering, we call C a **full rational** choice function.

From choice to revealed preference



Every choice function C induces two so-called **revealed preference relations**, namely

$$R_C := \varepsilon ; C ; \varepsilon^{\mathsf{T}}$$

$$R_C^* := (\varepsilon ; C \cap \overline{\varepsilon}) ; \varepsilon^{\mathsf{T}}$$

Suzumura writes this down as:

$$R_C^* = \bigcup_{S \in \mathcal{S}} \left[C(S) \times \{ S \setminus C(S) \} \right]$$

x is R_C^*-preferred to y if and only if x is chosen and y could have been chosen but was actually rejected from some $S \in \mathcal{S}$.
(Order reversed!)

From choice to revealed preference

$$C = \begin{array}{r|cccccccc} & \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} \\ \hline \{\} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \{1\} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \{2\} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \{1,2\} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \{3\} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \{1,3\} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \{2,3\} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \{1,2,3\} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}$$

$$R_C = \begin{array}{r|ccc} & 1 & 2 & 3 \\ \hline 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 2 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 3 & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{array} \qquad R_C^* = \begin{array}{r|ccc} & 1 & 2 & 3 \\ \hline 1 & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ 2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}$$

Axioms for cycle avoidance

- An H-**cycle** from x to x is given when $\left[R_C^* ; (R_C)^+\right]_{xx}$.
- An SH-**cycle** from x to x is given when $\left[R_C ; (R_C^*)^+\right]_{xx}$.

- **Houthakker's axiom** (HOA): $(R_C)^+ \subseteq \overline{R_C^*}^\top$,
- The **strong axiom** (SA): $(R_C^*)^+ \subseteq \overline{R_C}^\top$,
- The **weak axiom** (WA): $R_C^* \subseteq \overline{R_C}^\top$.

Axioms for cycle avoidance

- **strong congruence axiom** SCA:

  $$\varepsilon;C \cap {R_C^+}^\top;\varepsilon \subseteq \varepsilon$$

- **weak congruence axiom** WCA:

  $$\varepsilon;C \cap R_C^\top;\varepsilon \subseteq \varepsilon$$

The latter originally expressed as

$$\forall S \in \mathcal{S} : [x \in S \ \& \ \{\exists y \in C(S) : (x, y) \in R_C\}] \to x \in C(S)$$

Again without quantification over x!

Rational choice and revealed preference

Theorem

i) $HOA \iff SCA$

ii) $HOA \implies SA \implies WA$

iii) $WA \iff WCA$

iv) $WA \implies AA$

To prove (iii), we start from strict preference $R_C^* = (\varepsilon; C \cap \overline{\varepsilon}); \varepsilon^\mathsf{T}$ from WA:

$$R_C^\mathsf{T} \subseteq \overline{R_C^*} = \overline{(\varepsilon; C \cap \overline{\varepsilon}); \varepsilon^\mathsf{T}}$$

$$\iff (\varepsilon; C \cap \overline{\varepsilon}); \varepsilon^\mathsf{T} \subseteq \overline{R_C}^\mathsf{T}$$

$$\iff R_C^\mathsf{T}; \varepsilon \subseteq \overline{\varepsilon; C} \cup \varepsilon$$

$$\iff \varepsilon; C \cap R_C^\mathsf{T}; \varepsilon \subseteq \varepsilon, \qquad \text{i.e., WCA}$$

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Remarkable citations

George Boole's investigations on the laws of thought of 1854:

It would, perhaps, be premature to speculate here upon the question whether the methods of abstract science are likely at any future day to render service in the investigation of social problems at all commensurate with those which they have rendered in various departments of physical inquiry.

Remarkable citations

Garrett Birkhoff, 1967, Talk at the General Motors Research Laboratories

...for the two centuries preceding the development of the computer, broadly speaking, the main progress in applied mathematics was concerned with continuum analysis. Whereas over the last 20 years I think that the most conspicuous feature of the revolution that has taken place, and is continuing to take place, is a transition back from continuum mathematics towards digital mathematics; and one of the big questions, of course, is how far will this trend go or can it go?" In particular when considering the current trend of mathematizing also social considerations and concepts in the humanities, such a transition of known concepts from the continuous area down to the discrete world seems extremely promising.

Exzerpt of bibliography of trade union publication

BERGHAMMER, R., RUSINOWSKA, A., AND DE SWART, H. (2005) *Applying Relational Algebra and RELVIEW to Coalition Formation.* Public Choice Society. http://www.pubchoicesoc.org/papers2005/BerghammerRusinowskadeSwart.pdf

BRINK, C., KAHL, W., AND SCHMIDT, G. (EDS.) (1997) *Relational Methods in Computer Science.* Berlin, Springer.

DEEMEN, A. VAN (1997) *Coalition Formation and Social Choice.* Kluwer.

RUSINOWSKA, A., DE SWART, H., AND VAN DER RIJT, J.W. (2005) A new model of coalition formation. *Social Choice and Welfare,* 24, 129–154.

SCHMIDT, G., AND STRÖHLEIN, T. (1993) *Relations and Graphs, Discrete Mathematics for Computer Scientists.* Berlin, Springer.

SWART, H. DE, ORLOWSKA, E., SCHMIDT, G., AND ROUBENS, M. (EDS.) (2003) *Theory and Applications of Relational Structures as Knowledge Instruments.* Berlin, Springer.

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History of relations

Relations were being developed at a time when

- formal semantics was not yet known

  language — interpretation — typing — unification
- the idea that several models of a theory may exist, was close to being completely unknown

  (non-Euclidian geometry: Bolyai, Lobatschevskij $\approx$ 1840)
- one was still bound to handle the following in the respective natural language, namely in English, German, Latin, Greek, Japanese, Russian, Arabic ...!

quantification	$\forall, \exists$
conversion	R^{T}
composition	$A ; B$

  but also „brother“, „father“, „uncle“
- the concept of a matrix had not yet been coined

  (Cayley, Sylvester 1850's)

History of relations

- Aristotle (384–322 b.C.): Syllogisms were not suited for:

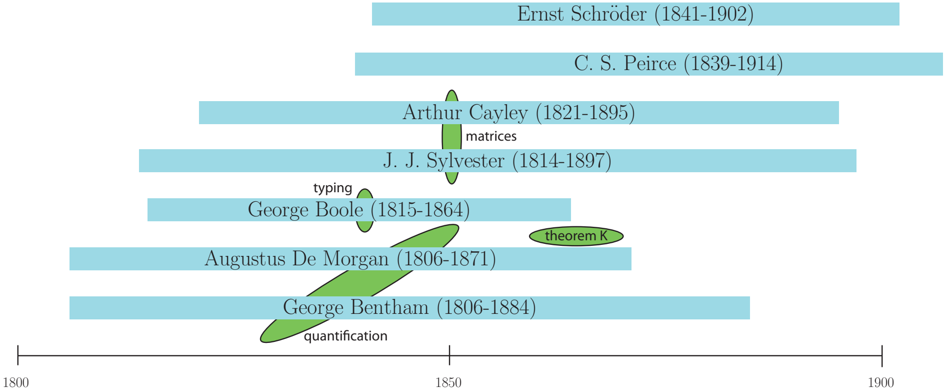
$$\begin{array}{l}\text{head of a horse}\\ \text{horse is an animal}\end{array} \quad\Longrightarrow\quad \text{head of an animal}$$

- Peter Abaelard (1079–1142)
- William of Ockham (1287–1347)
- Gottfried Wilhelm Leibniz (1646–1716)

...

- Charles Sanders Peirce (1839–1914)
- Ernst Schröder (1841–1902)
- Alfred Tarski (1902–1983)

History of relations



Thank you for your attention!