Extension properties of Boolean contact algebras

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Boolean contact algebras

A Boolean contact algebra $\langle B, \mathscr{C} \rangle$ is a Boolean algebra B together with a binary relation $\mathscr{C}$ on B which satisfies for all $x, y, z \in B$

C_0. $0(-\mathscr{C})x$

C_1. $x \neq 0$ implies $x \mathscr{C} x$ (domain reflexivity)

C_2. $x \mathscr{C} y$ implies $y \mathscr{C} x$ (symmetry)

C_3. $x \mathscr{C} y$ and $y \leq z$ implies $x \mathscr{C} z$. (monotonicity)

C_4. $x \mathscr{C} (y + z)$ implies ($x \mathscr{C} y$ or $x \mathscr{C} z$) (distributivity)

A BCA is *connected* if

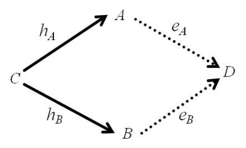
- $x \neq 0$ and $x \neq 1$ implies $x \mathscr{C} -x$ (connectivity)

Hereditary and extension properties

A class **K** of $\mathscr{L}$ – structures has the

1. *Hereditary property* (HP) if **K** is closed under substructures.
2. *Joint embedding property* (JEP) if for any $A, B \in \mathbf{K}$, there are some $C \in \mathbf{K}$ and embeddings $e_A : A \to C,\ e_B : B \to C$.
3. *Amalgamation property* (AP) if for all $A, B, C \in \mathbf{K}$ such that C is (isomorphic to) a common substructure of A and B, say, with embeddings $h_B : C \hookrightarrow B$ and $h_A : C \hookrightarrow A$, there are some $D \in \mathbf{K}$ and embeddings $e_A : A \hookrightarrow D$ and $e_B : B \hookrightarrow D$ such that $e_A \circ h_A = e_B \circ h_B$.

Figure: Amalgamation property



Fraïssé limits

In this work, we prove that the class of BCAs has HP, JEP, and AP. Thus, by Fraïssé's result,

Theorem

There is a countable homogeneous and ω – categorical BCA $\mathfrak{B} = \langle B, \mathscr{C} \rangle$, which is universal in the sense that each countable BCA is isomorphic to a substructure of $\mathfrak{B}$.

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In a companion paper, we show

Theorem

The relation algebra $\mathfrak{A}$ *generated by* $\mathscr{C}$ *on* $\mathfrak{B}$ *is finite.*

Qualitative spatial reasoning

- Investigates properties of relations – “part–of”, “contact”

S. Leśniewski
1886–1939

- Mereology (Leśniewski, 1915)
- Spatio - temporal relations (Nicod, 1920)
- Pointless geometry (De Laguna, 1922, Whitehead, 1929)
- Geometry of solids (Tarski, 1927)
- Region Connection Calculus (Randell, Cui, and Cohn, 1992)

- No knowledge about “points” is necessary.
- The relational calculus is “pointless”.

Standard contact structures

- The regular closed sets of a topological space X form a complete Boolean algebra under the operations

  $$a + b = a \cup b,\ a \cdot b = \mathrm{cl}(\mathrm{int}(a \cap b)),\ a^* = \mathrm{cl}(X \setminus a),\ 0 = \emptyset,\ 1 = X.$$

  Write $\mathrm{RegCl}(X)$ for this Boolean algebra. Observe that it is possible that $a \cdot b = 0$, but $a \cap b \neq \emptyset$.

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  Write $\mathrm{RegCl}(X)$ for this Boolean algebra. Observe that it is possible that $a \cdot b = 0$, but $a \cap b \neq \emptyset$.
- The standard contact relation C_τ on $\mathrm{RegCl}(X)$ is defined as, for all regular closed sets a, b

  $$a C_\tau b \iff a \cap b \neq \emptyset.$$

Basic facts on BCAs

- There is only one contact relation on $\mathbf{2} = \{0, 1\}$.
- Let B be a Boolean algebra.
  - The smallest contact relation on B is given by $x \mathscr{C}_{\mathsf{min}} y \iff x \cdot y \neq 0$. $\mathscr{C}_{\mathsf{min}}$ is called the *overlap relation*.
  - The largest contact relation on B is $\mathscr{C}_{\mathsf{max}} = B^+ \times B^+$.

Finite BCAs and graphs

Let B be a finite Boolean algebra, and $\mathrm{At}(B)$ its atom set.

*a reflexive and symmetric relation on X = a *graph* on X*

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There is a bijective order preserving correspondence between the contact relations on B and the graphs on $\mathrm{At}(B)$.

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- If $\mathscr{C}$ is a contact relation on B, let

$$R_{\mathscr{C}} = \{\langle a, b\rangle \in \mathrm{At}(B)^2 : \uparrow a \times \uparrow b \subseteq \mathscr{C}\}$$

  Note: for atoms a, b, $\langle a, b\rangle \in \mathscr{C}$ iff $\uparrow a \times \uparrow b \subseteq \mathscr{C}$.
- If R is a graph on $\mathrm{At}(B)$, set

$$\mathscr{C}_R = \bigcup\{\uparrow a \times \uparrow b : \langle a, b\rangle \in R\}$$

- $\mathscr{C}_{R_{\mathscr{C}}} = \mathscr{C}$, $R_{\mathscr{C}_R} = R$.

What kind of graph morphisms correspond to BCA embeddings?

Strong graph homomorphism (sgh)

Let A, C be two graphs. A mapping $f : (A, R_A) \to (C, R_C)$ is a sgh if R_C is the image of R_A under f, i.e.

- f is an onto graph homomorphism
- any edge in C is the image of some edge in A under f.

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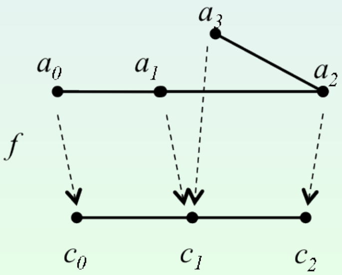
Example: Suppose $A = \{a_0, a_1, a_2, a_3\}$ and $C = \{c_0, c_1, c_2\}$ are as below. Then $f : A \to C$ is a strong graph homomorphism.



BCA embeddings and strong graph homomorphisms

Theorem

Let $\langle A, \mathscr{C}_A \rangle$ *and* $\langle B, \mathscr{C}_B \rangle$ *be two finite BCAs.*

- *If* $e : A \to B$ *is a BCA embedding, then* $u : \mathrm{At}(B) \to \mathrm{At}(C)$ *is a strong graph homomorphism, where*

$$u(x) = \bigwedge \{a \in A : \ e(a) \geq x\}.$$

- *If* $u : \mathrm{At}(B) \to \mathrm{At}(A)$ *is a strong graph homomorphism, then* $e : A \to B$ *is a BCA embedding, where*

$$e(a) = \bigvee \{x \in \mathrm{At}(B) : \ u(x) \leq a\}.$$

Duality for BCAs

Let B be a Boolean algebra and $\mathrm{Ult}(B)$ its Stone space.

Theorem (Düntsch & Winter, RelMiCS 10)

There is a bijective order preserving correspondence between the contact relations on B and the graphs on $\mathrm{Ult}(B)$ which are closed in the product topology of $\mathrm{Ult}(B)$.

- If R is a graph on $\mathrm{Ult}(B)$, set $\mathscr{C}_R = \bigcup\{F \times G : \langle F, G\rangle \in R\}$.
- If $\mathscr{C}$ is a contact relation on B, let $R_{\mathscr{C}} = \{\langle F, G\rangle \in \mathrm{Ult}(B)^2 : F \times G \subseteq \mathscr{C}\}$.
- $\mathscr{C}_{R_{\mathscr{C}}} = \mathscr{C}$, $R_{\mathscr{C}_R} = R$.

Duality for maps

Let A, B be two Boolean algebras and $e : A \to B$ a Boolean embedding. The dual of e is the map $u : \mathrm{Ult}(B) \twoheadrightarrow \mathrm{Ult}(A)$ defined by $u(F) = e^{-1}(F)$. If e is an inclusion, then $u(F) = F \cap A$.

Theorem

- *Let $\langle A, \mathscr{C}_A\rangle$ and $\langle B, \mathscr{C}_B\rangle$ be BCAs with dual relations R_A and R_B, and suppose that A is a (Boolean) subalgebra of B. Then, $\mathscr{C}_A = \mathscr{C}_B \restriction A$ if and only if*

$$R_A = \{\langle F \cap A, G \cap A\rangle : F, G \in \mathrm{Ult}(B), \langle F, G\rangle \in R_B\}.$$

- *e is a BCA embedding iff R_B is the image of R_A under u.*

BCAs and HP, JEP, AP

Let $\mathbf{K}^0$ be the class of all BCAs. Then

- $\mathbf{K}^0$ has HP, since the axioms of BCAs are universal.
- JEP is a special case of AP, since $\mathbf{2} = \{0, 1\}$, the smallest BCA, can be embedded into any BCA.

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So we need only prove

Theorem

The class $\mathbf{K}^0$ of BCAs has the amalgamation property.

Amalgamation property and BCAs: The finite case

sgh = strong graph homomorphism

Theorem

Suppose A, B, C are finite graphs and $f : A \to C$ and $g : B \to C$ are sghs. Then there are a graph D, and sghs $e_A : D \to A$ and $e_B : D \to B$ s.t. the following diagram is commutative:

$$
\begin{tikzcd}
 & A \arrow[dl, "f"'] & \\
C & & D \arrow[ul, dotted, "e_A"'] \arrow[dl, dotted, "e_B"] \\
 & B \arrow[ul, "g"] &
\end{tikzcd}
$$

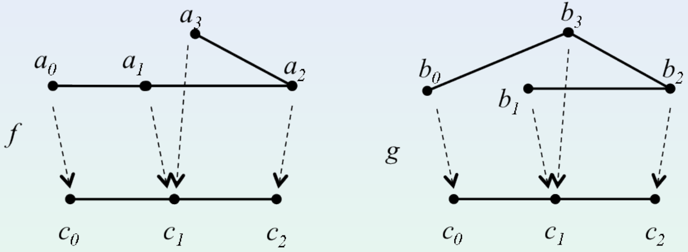
Construction of D: The finite case

Let

$$D = \{\langle a, b\rangle \in A \times B : \ f(a) = g(b)\}$$
$$\langle\langle a, b\rangle, \langle a', b'\rangle\rangle \in R_D \iff \langle a, a'\rangle \in R_A, \langle b, b'\rangle \in R_B$$
$$e_A(\langle a, b\rangle) = a$$
$$e_B(\langle a, b\rangle) = b$$

Then e_A and e_B are sghs and $f \circ e_A = g \circ e_B$.

An example



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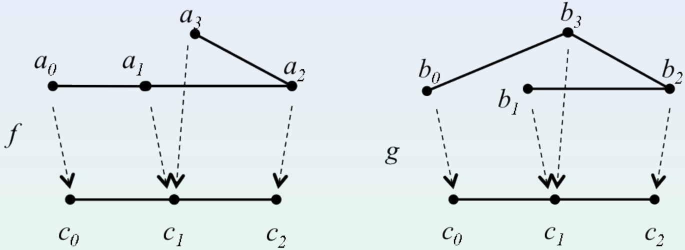

$$D = \begin{tikzcd} \langle a_1, b_1 \rangle \arrow[r, dash] & \langle a_2, b_2 \rangle \arrow[r, dash] \arrow[dl, dash] \arrow[dr, dash] & \langle a_3, b_1 \rangle \\ \langle a_1, b_3 \rangle \arrow[r, dash] & \langle a_0, b_0 \rangle & \langle a_3, b_3 \rangle \end{tikzcd}$$

Construction of D: The infinite case

Suppose that $\mathfrak{A} = \langle A, \mathscr{C}_A \rangle$ and $\mathfrak{B} = \langle B, \mathscr{C}_B \rangle$ are two BCAs and $\mathfrak{C} = \langle C, \mathscr{C}_C \rangle$ is isomorphic to a common substructure of $\mathfrak{A}$ and $\mathfrak{B}$. Let D be the Boolean amalgamated free product of A and B over C, w.l.o.g. $A, B \leq D$. Define $R \subseteq \mathrm{Ult}(D) \times \mathrm{Ult}(D)$ by

$$\langle H, H' \rangle \in R \Longleftrightarrow \langle H \cap A, H' \cap A \rangle \in R_{\mathscr{C}_A} \text{ and } \langle H \cap B, H' \cap B \rangle \in R_{\mathscr{C}_B}.$$

- R is reflexive, symmetric and closed.
- The dual of R is a contact relation on D extending both $\mathscr{C}_A$ and $\mathscr{C}_B$.

Theorem

The class K^0 of BCAs has the amalgamation property.

Theorem

The class K^1 *of connected Boolean contact algebras does not have the amalgamation property.*

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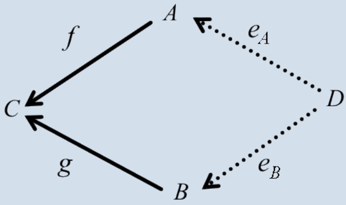
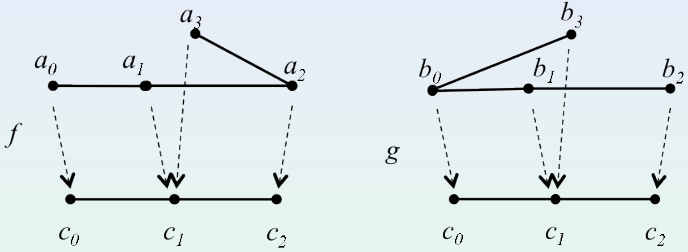
Proof.

We prove this by showing that, for some sghs $f : A \to C$ and $g : B \to C$ any amalgamation D of A, B over C in K^0 is not a connected graph



$\square$

A counter example



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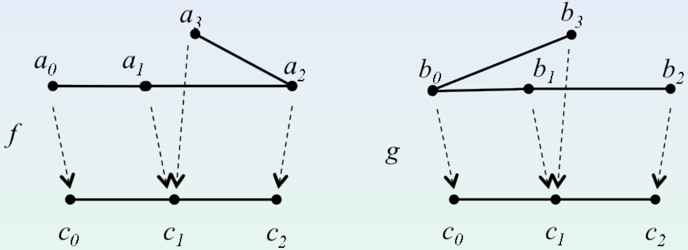

$$D = \begin{tikzcd}
\langle a_0, b_0\rangle \arrow[r, dash] \arrow[d, dash] & \langle a_1, b_1\rangle \arrow[d, dash] & \langle a_3, b_3\rangle \\
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\end{tikzcd}$$

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In a sense, $\mathfrak{B}$ (or the corresponding closed graph on $\mathrm{Ult}(B)$) can be regarded as another 'limit' of finite graphs.

Mnogo blagodarya
Thank you
Dziękuję
Danke
Merci