Verification of Uninterpreted and Partially Interpreted Programs

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Introduction

Program Verification

Program verification is undecidable, in general.

However, decidable classes do exist:

- Programs without loops or recursion (straight-line)
- Programs working over finite domains (Boolean programs)
- Models like Petri Nets - not natural for modeling programs

Today : Decidable verification for programs with loops/recursion while working over infinite domains.

Uninterpreted Programs

What are Uninterpreted Programs?

- Programs over an uninterpreted vocabulary
  - Constant, function and relation symbols are *completely uninterpreted*.
- Work over arbitrary data models
  - Data models provide interpretations to symbols in the program.
- Satisfy ϕ if ϕ holds in *all* data models

Uninterpreted Programs: Syntax

Fix a finite set V of program variables.
Fix a first order vocabulary $\Sigma = (\mathcal{C}, \mathcal{F}, \mathcal{R})$.

Program Syntax

$$
\begin{aligned}
\langle \text{stmt} \rangle ::= \; & \textbf{skip} \mid x := c \mid x := y \mid x := f(\mathbf{z}) \\
& \mid \textbf{if} \, (\langle \text{cond} \rangle) \, \textbf{then} \, \langle \text{stmt} \rangle \, \textbf{else} \, \langle \text{stmt} \rangle \mid \textbf{while} \, (\langle \text{cond} \rangle) \, \langle \text{stmt} \rangle \\
& \mid \textbf{assume} \, (\langle \text{cond} \rangle) \mid \langle \text{stmt} \rangle \, ; \, \langle \text{stmt} \rangle
\end{aligned}
$$

$$
\begin{aligned}
\langle \text{cond} \rangle ::= \; & \texttt{true} \mid x = y \mid x = c \mid c = d \mid R(\mathbf{z}) \\
& \mid \langle \text{cond} \rangle \vee \langle \text{cond} \rangle \mid \neg \langle \text{cond} \rangle
\end{aligned}
$$

where, $x, y, \mathbf{z} \in V$, $c \in \mathcal{C}$, $f \in \mathcal{F}$ and $R \in \mathcal{R}$.

Example

```
assume (T ≠ F);
b := F;
while (x ≠ y) {
    d := key(x);
    if (d = k) then {
        b := T;
        r := x;
    }
    x := n(x);
}
```

- Searches for an element with key `k` in a list starting at `x` and ending at `y`.
- `T` and `F` are uninterpreted constants
- `key` and `n` are uninterpreted functions

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Uninterpreted Programs: Executions

Executions are finite sequences over the following alphabet

$$\Pi = \left\{ \begin{array}{l} \text{“}x := y\text{”}, \text{“}x := f(\mathbf{z})\text{”}, \\ \text{“}\mathbf{assume}(x = y)\text{”}, \text{“}\mathbf{assume}(x \neq y)\text{”}, \\ \text{“}\mathbf{assume}(R(\mathbf{z}))\text{”}, \text{“}\mathbf{assume}(\neg R(\mathbf{z}))\text{”} \end{array} \middle| \begin{array}{l} x, y, \mathbf{z} \in V, \\ f \in \mathcal{F}, R \in \mathcal{R} \end{array} \right\}$$

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Set of executions is a regular language defined inductively:

$$\begin{array}{lcl}
\mathsf{Exec}(\mathbf{skip}) & = & \{\epsilon\} \\
\mathsf{Exec}(x := y) & = & \{\text{“}x := y\text{”}\} \\
\mathsf{Exec}(x := f(\mathbf{z})) & = & \{\text{“}x := f(\mathbf{z})\text{”}\} \\
\mathsf{Exec}(\mathbf{assume}(c)) & = & \{\text{“}\mathbf{assume}(c)\text{”}\} \\
\mathsf{Exec}(\mathbf{if}\ c\ \mathbf{then}\ s_1\ \mathbf{else}\ s_2) & = & \begin{array}{l} \{\text{“}\mathbf{assume}(c)\text{”}\} \cdot \mathsf{Exec}(s_1) \\ \cup \{\text{“}\mathbf{assume}(\neg c)\text{”}\} \cdot \mathsf{Exec}(s_2) \end{array} \\
\mathsf{Exec}(s_1; s_2) & = & \mathsf{Exec}(s_1) \cdot \mathsf{Exec}(s_2) \\
\mathsf{Exec}(\mathbf{while}\ c\ \{s\}) & = & \Big(\{\text{“}\mathbf{assume}(c)\text{”}\} \cdot \mathsf{Exec}(s)\Big)^* \cdot \{\text{“}\mathbf{assume}(\neg c)\text{”}\}
\end{array}$$

Semantics given by a first order structure $\mathtt{M} = (\mathcal{U}_{\mathtt{M}}, [\![]\!]_{\mathtt{M}})$ on Σ.

Definition (Values of Variables)

$$
\begin{array}{rcll}
\mathsf{val}_{\mathtt{M}}(\epsilon, x) & = & [\![\widehat{x}]\!]_{\mathtt{M}} & \text{for every } x \in V \\
\mathsf{val}_{\mathtt{M}}(\rho\cdot\text{``}x := y\text{''}, z) & = & \mathsf{val}_{\mathtt{M}}(\rho, y) & \text{if } z \text{ is } x \\
\mathsf{val}_{\mathtt{M}}(\rho\cdot\text{``}x := f(z_1, \ldots)\text{''}, y) & = & [\![f]\!]_{\mathtt{M}}(\mathsf{val}_{\mathtt{M}}(\rho, z_1), \ldots) & \text{if } y \text{ is } x \\
\mathsf{val}_{\mathtt{M}}(\rho\cdot a, x) & = & \mathsf{val}_{\mathtt{M}}(\rho, x) & \text{otherwise}
\end{array}
$$

Uninterpreted Programs: Semantics

Semantics given by a first order structure $\mathtt{M} = (\mathcal{U}_{\mathtt{M}}, [\![]\!]_{\mathtt{M}})$ on Σ.

Definition (Feasibility of Execution)

An execution ρ is feasible in $\mathtt{M}$ if for every prefix $\sigma' = \sigma \cdot$ "**assume**(c)" of ρ, we have

1. $\mathsf{val}_{\mathtt{M}}(\sigma, x) = \mathsf{val}_{\mathtt{M}}(\sigma, y)$ if c is $(x = y)$,
2. $\mathsf{val}_{\mathtt{M}}(\sigma, x) \neq \mathsf{val}_{\mathtt{M}}(\sigma, y)$ if c is $(x \neq y)$,
3. $(\mathsf{val}_{\mathtt{M}}(\sigma, z_1), \ldots, \mathsf{val}_{\mathtt{M}}(\sigma, z_r)) \in [\![R]\!]_{\mathtt{M}}$ if c is $R(z_1, \ldots, z_r)$, and
4. $(\mathsf{val}_{\mathtt{M}}(\sigma, z_1), \ldots, \mathsf{val}_{\mathtt{M}}(\sigma, z_r)) \notin [\![R]\!]_{\mathtt{M}}$ if c is $\neg R(z_1, \ldots, z_r)$.

Uninterpreted Programs: Verification

Definition (Verification of Uninterpreted Programs)

Let $P \in \langle \text{stmt} \rangle$ be an uninterpreted program and let φ be an assertion in the following grammar.

$$\varphi ::= \texttt{true} \mid x = y \mid R(\mathbf{z}) \mid \varphi \vee \varphi \mid \neg\varphi$$

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Theorem [1, 3]

Verification of uninterpreted programs is undecidable.

Coherence

How do we verify a single execution?

Execution ρ

$$
\begin{gathered}
\textbf{assume}(\mathtt{T} \neq \mathtt{F}) \\
\mathtt{b} := \mathtt{F} \\
\textbf{assume}(\mathtt{x} \neq \mathtt{y}) \\
\mathtt{d} := \mathtt{key(x)} \\
\textbf{assume}(\mathtt{d} = \mathtt{k}) \\
\mathtt{b} := \mathtt{T} \\
\mathtt{r} := \mathtt{x} \\
\mathtt{x} := \mathtt{n(x)} \\
\textbf{assume}(\mathtt{x} = \mathtt{y})
\end{gathered}
$$

$\varphi \equiv \mathtt{b{=}T} \Rightarrow \mathtt{key(r){=}k}$

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$VC(\rho, \varphi)$

$$\begin{array}{lc}
 & \mathtt{T} \neq \mathtt{F} \\
\wedge & \mathtt{b}_1 = \mathtt{F} \\
\wedge & \mathtt{x}_0 \neq \mathtt{y}_0 \\
\wedge & \mathtt{d}_1 = \mathtt{key}(\mathtt{x}_0) \\
\wedge & \mathtt{d}_1 = \mathtt{k}_0 \\
\wedge & \mathtt{b}_2 = \mathtt{T} \\
\wedge & \mathtt{r}_1 = \mathtt{x}_0 \\
\wedge & \mathtt{x}_1 = \mathtt{n}(\mathtt{x}_0) \\
\wedge & \mathtt{x}_1 = \mathtt{y}_0 \\
 & \\
\Rightarrow & (\mathtt{b}_2 = \mathtt{T} \Rightarrow \mathtt{key}(\mathtt{r}_1) = \mathtt{k}_0)
\end{array}$$

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\wedge & \mathtt{b}_2 = \mathtt{T} \\
\wedge & \mathtt{r}_1 = \mathtt{x}_0 \\
\wedge & \mathtt{x}_1 = \mathtt{n}(\mathtt{x}_0) \\
\wedge & \mathtt{x}_1 = \mathtt{y}_0 \\
 & \\
\Rightarrow & (\mathtt{b}_2 = \mathtt{T} \Rightarrow \mathtt{key}(\mathtt{r}_1) = \mathtt{k}_0)
\end{array}
$$

φ holds in every $\mathtt{M}$ in which ρ is feasible

iff

$VC(\rho, \varphi)$ is valid in T_{EUF}

How do we verify a single execution?

- Verification of *a single execution* can be reduced to checking validity of a quantifier-free formula in T_{EUF}.

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  - Polynomial time when φ is a single atom.
- But programs have infinitely many executions.
- How do we recover decidability?
- Coherence to the rescue!
  - Allows congruence closure to be performed in a *streaming* fashion.

Congruence Closure

Congruence on Ground Terms

Let $\Sigma = (\mathcal{C}, \mathcal{F})$ be a FO-vocabulary. Let $t_1, t_1', t_2, \ldots, t_k, t_k'$ be ground terms on Σ and let $f \in \mathcal{F}$ be a k-ary function. Then,

$$\frac{t_1 = t_1' \qquad t_2 = t_2' \qquad \ldots \qquad t_k = t_k'}{f(t_1, t_2, \ldots, t_k) = f(t_1', t_2', \ldots, t_k')}$$

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Congruence on Ground Terms

Let $\Sigma = (\mathcal{C}, \mathcal{F})$ be a FO-vocabulary. Let $t_1, t'_1, t_2, \ldots, t_k, t'_k$ be ground terms on Σ and let $f \in \mathcal{F}$ be a k-ary function. Then,

$$\frac{t_1 = t'_1 \qquad t_2 = t'_2 \qquad \ldots \qquad t_k = t'_k}{f(t_1, t_2, \ldots, t_k) = f(t'_1, t'_2, \ldots, t'_k)}$$

Interpretation

In every FO structure $\mathtt{M}$,

$$\begin{array}{rc} \text{if} & [\![t_1]\!]_{\mathtt{M}} = [\![t'_1]\!]_{\mathtt{M}},\ [\![t_2]\!]_{\mathtt{M}} = [\![t'_2]\!]_{\mathtt{M}},\ \ldots,\ \text{and } [\![t_k]\!]_{\mathtt{M}} = [\![t'_k]\!]_{\mathtt{M}} \\ \text{then} & [\![f(t_1, t_2, \ldots, t_k)]\!]_{\mathtt{M}} = [\![f(t'_1, t'_2, \ldots, t'_k)]\!]_{\mathtt{M}} \end{array}$$

Congruence Closure on Executions

$$\textbf{assume}(x = y) \longrightarrow x_1 := f(x) \longrightarrow y_1 := f(y)$$

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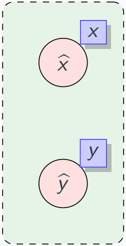
Initially



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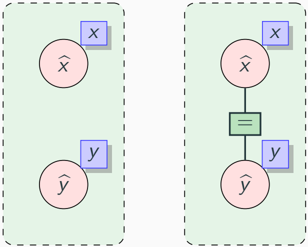

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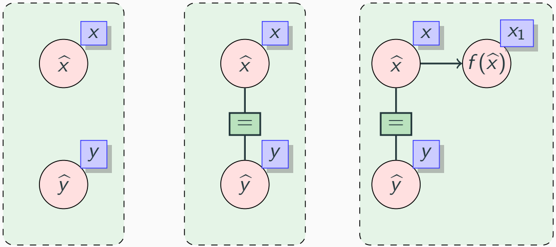

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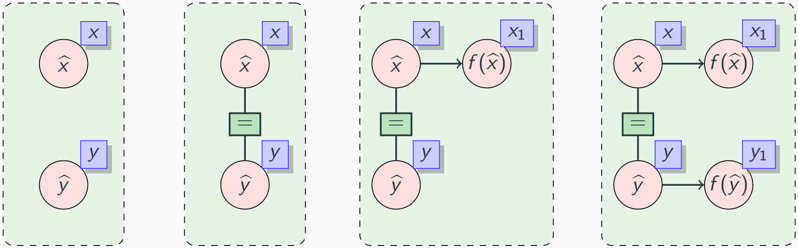

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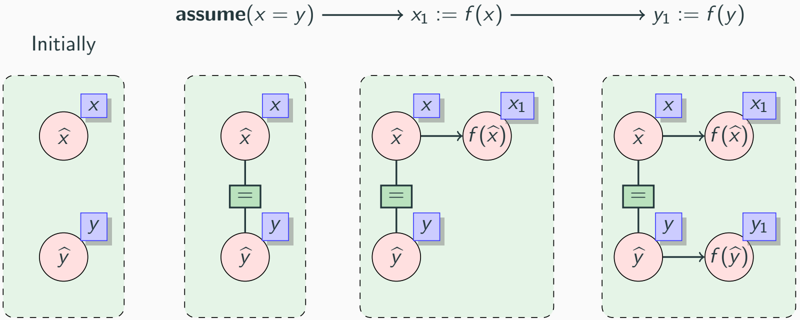

$\varphi : x_1 = y_1$

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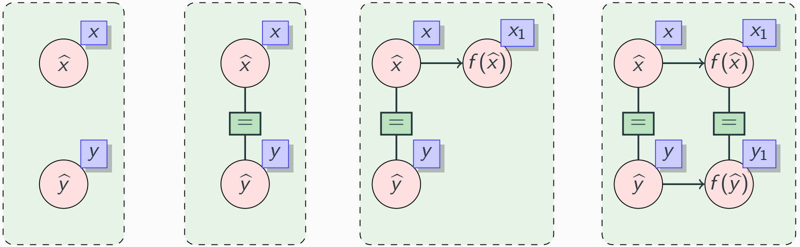

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φ holds
after the execution

Congruence Closure on Executions

$$\textbf{assume}(x = y) \longrightarrow x := f(x) \overset{n \text{ times}}{\dashrightarrow} x := f(x) \xrightarrow{\qquad\qquad} y := f(y) \overset{n \text{ times}}{\dashrightarrow} y := f(y)$$

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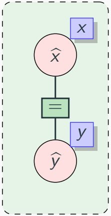

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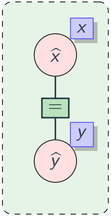

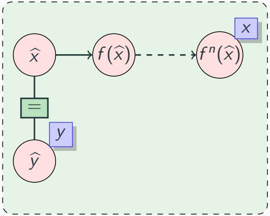

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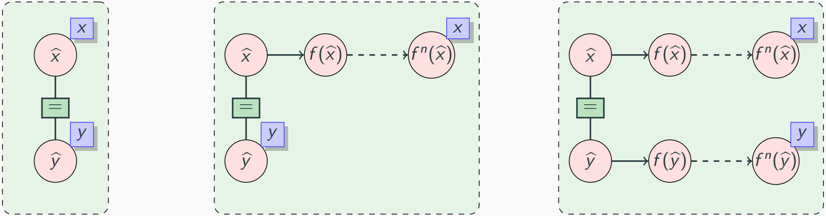

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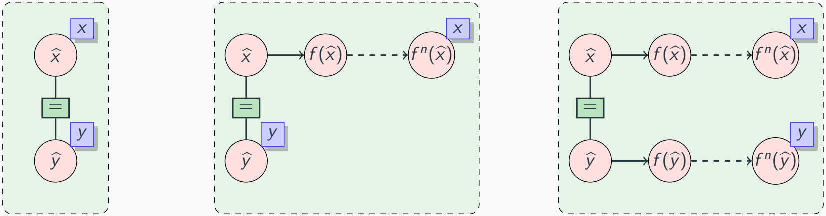

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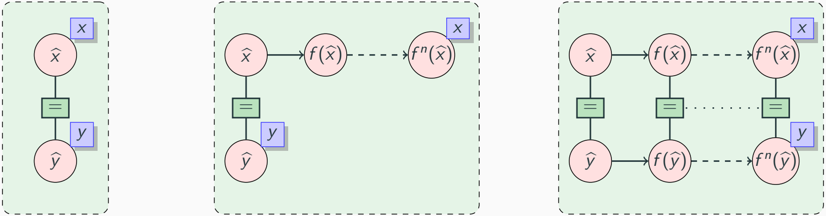

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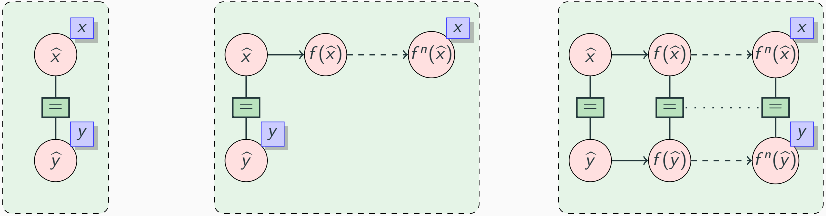

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Unbounded memory required to infer equality relationships in a streaming setting.

Congruence Closure on Executions

$$x_1 := f(x) \rightarrow y_1 := f(y) \overset{n \text{ times}}{\dashrightarrow} x_1 := f(x) \longrightarrow y_1 := f(y_1) \longrightarrow \textbf{assume}(x = y)$$

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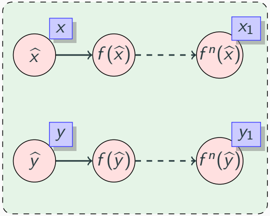
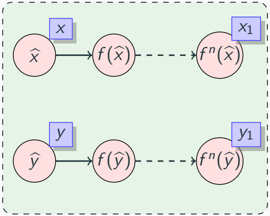
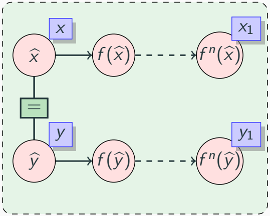


Congruence Closure on Executions

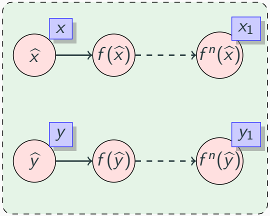
$$x_1 := f(x) \to y_1 := f(y) \overset{n \text{ times}}{\dashrightarrow} x_1 := f(x) \longrightarrow y_1 := f(y_1) \longrightarrow \textbf{assume}(x = y)$$

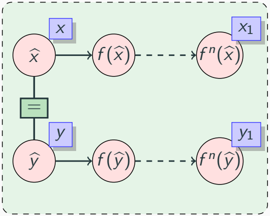




Congruence Closure on Executions

$$x_1 := f(x) \rightarrow y_1 := f(y) \overset{n \text{ times}}{\dashrightarrow} x_1 := f(x) \longrightarrow y_1 := f(y_1) \longrightarrow \textbf{assume}(x = y)$$





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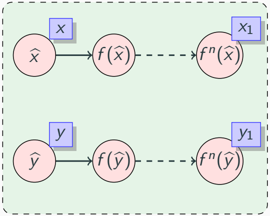

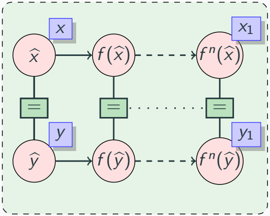

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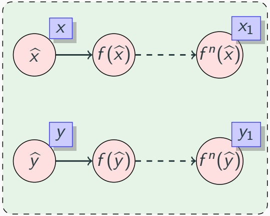

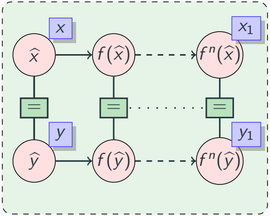

$\varphi :\ x_1 = y_1$

φ holds
after the execution

Again, unbounded memory required to infer equality relationships in a streaming setting.

Algebraic View of Executions

Terms Computed

$$\begin{aligned}
\mathrm{Term}(\epsilon, x) &= \widehat{x} && \text{for every } x \in V \\
\mathrm{Term}(\rho \cdot \text{“}x := y\text{”}, z) &= \mathrm{Term}(\rho, y) && \text{if } z \text{ is } x \\
\mathrm{Term}(\rho \cdot \text{“}x := f(z_1, \ldots)\text{”}, y) &= f(\mathrm{Term}(\rho, z_1), \ldots) && \text{if } y \text{ is } x \\
\mathrm{Term}(\rho \cdot a, x) &= \mathrm{Term}(\rho, x) && \text{otherwise}
\end{aligned}$$

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$$

Equalities

$$
\begin{aligned}
\alpha(\varepsilon) &= \varnothing \\
\alpha(\rho \cdot \text{“}\textbf{assume}(x = y)\text{”}) &= \alpha(\rho) \cup \{(\text{Term}(\rho, x), \text{Term}(\rho, y))\} \\
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Disequalities

$$
\begin{aligned}
\beta(\varepsilon) &= \varnothing \\
\beta(\rho \cdot \text{“}\textbf{assume}(x \neq y)\text{”}) &= \beta(\rho) \cup \{(\mathrm{Term}(\rho, x), \mathrm{Term}(\rho, y))\} \\
\beta(\rho \cdot a) &= \beta(\rho) \quad \text{otherwise}
\end{aligned}
$$

Coherence

An execution is coherent if it is memoizing and has early assumes.

Coherence = Memoizing + Early Assumes

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An execution is coherent if it is memoizing and has early assumes.

Coherence = Memoizing + Early Assumes

Remember terms
until re-computation

Equality assumes
occur before
superterms are forgotten

Coherence: Memoizing

Definition (Memoizing Execution)
An execution ρ is memoizing if for every prefix of ρ of the form

$$\sigma' = \sigma \cdot \text{“}x := f(y_1, \ldots, y_r)\text{”}$$

we have the following.

If there is a term $t \in \mathrm{ComputedTerms}(\sigma)$ such that $t \cong_{\alpha(\sigma)} \mathrm{Term}(\sigma', x)$, then there is a variable $z \in V$ such that $\mathrm{Term}(\sigma, z) \cong_{\alpha(\sigma)} \mathrm{Term}(\sigma', x)$.

Here,

- $\mathrm{ComputedTerms}(\sigma) = \{\mathrm{Term}(\pi, v) \mid v \in V, \pi \text{ is a prefix of } \sigma\}$,
- $\cong_{\alpha(\rho)}$ is the smallest congruence induced by $\alpha(\rho)$.

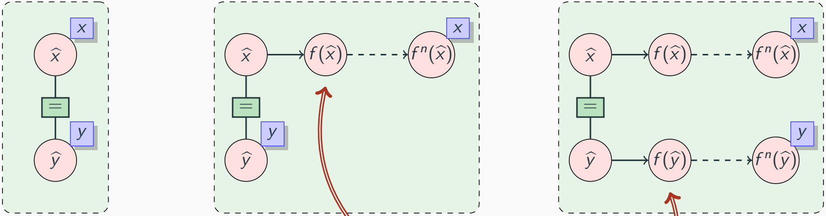
Coherence: Memoizing

```
assume (T ≠ F);
b := F;
while (x ≠ y) {
   d := key(x);
   if (d = k) then {
      b := T;
      r := x;
   }
   x := n(x);
}
```

- All executions of this program are *vacuously* memoizing.
- No term is recomputed.

Example exeuction: Non Memoizing

$$\textbf{assume}(x = y) \longrightarrow x := f(x) \overset{n \text{ times}}{\dashrightarrow} x := f(x) \longrightarrow y := f(y) \overset{n \text{ times}}{\dashrightarrow} y := f(y)$$



Re-computation of terms deemed equivalent by $\widehat{x} = \widehat{y}$.
The older term $f(\widehat{x})$ has been dropped.

NOT a memoizing execution

Example exeuction: Memoizing

$$\textbf{assume}(x = y) \longrightarrow x := f(x) \longrightarrow y := f(y) \overset{n \text{ times}}{\dashrightarrow} x := f(x) \longrightarrow y := f(y)$$

Re-computation happens in tandem
(at least one older equivalent terms is available
in some variable at the time of re-computation)

✓ memoizing execution

Definition (Early Assumes)

An execution ρ is said to have early assumes if for every prefix of ρ of the form

$$\sigma' = \sigma \cdot \text{“}\textbf{assume}(x = y)\text{”}$$

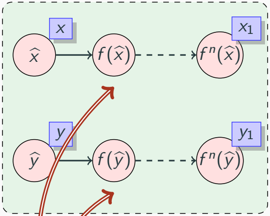
we have the following.

If there is a term $s \in \mathrm{ComputedTerms}(\sigma)$ such that s is a $\alpha(\sigma)$-*superterm* of either $\mathrm{Term}(\sigma, x)$ or $\mathrm{Term}(\sigma, y)$, then there is a variable $z \in V$ such that $\mathrm{Term}(\sigma, z) \cong_{\alpha(\sigma)} s$.

Here, t_1 is a $\alpha(\sigma)$-superterm of t_2 if there are terms t_1' and t_2' such that t_1' is a superterm of t_2', $t_1 \cong_{\alpha(\sigma)} t_1'$ and $t_2 \cong_{\alpha(\sigma)} t_2'$.

Example exeuction: Violation of Early Assumes

$$x_1 := f(x) \to y_1 := f(y) \overset{n \text{ times}}{\dashrightarrow} x_1 := f(x) \longrightarrow y_1 := f(y_1) \longrightarrow \textbf{assume}(x = y)$$



Superterms of $\widehat{x}$ and $\widehat{y}$
dropped before equality assume.

Does NOT satisfy early assumes

Example exeuction: Early Assumes

$$\textbf{assume}(x = y) \longrightarrow x := f(x) \longrightarrow y := f(y) \xrightarrow{n \text{ times}} x := f(x) \longrightarrow y := f(y)$$

✓ Early Assume

Coherence

```
assume (T ≠ F);
b := F;
while (x ≠ y) {
   d := key(x);
   if (d = k) then {
      b := T;
      r := x;
   }
   x := n(x);
}
```

- In every execution, equality assume **assume**($x = y$) occurs on terms without any superterms.
- All executions are coherent!

Coherent Programs and their Verification

An uninterpreted program $P \in \langle \mathsf{stmt} \rangle$ is coherent if all executions of P are coherent.

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Decidability of Verification of Coherent Programs [1]

Verification of uninterpreted coherent programs is PSPACE-complete.

Proof.

- Regular language $L^{\varphi}_{\text{coherent}}$ such that for any coherent execution ρ,

$$\rho \in L^{\varphi}_{\text{coherent}} \text{ iff } \rho \models \varphi$$

- The question $\mathsf{Exec}(P) \subseteq L^{\varphi}_{\text{coherent}}$ is decidable.

Regularity of Feasible Coherent Executions

- $P \models \varphi$ iff $P^{\neg\varphi} \models \texttt{false}$, where $P^{\neg\varphi} = P; \textbf{assume}(\neg\varphi)$
- Regular language $L_{\text{coh-feas}}$ such that for any coherent execution ρ,

$$\rho \in L_{\text{coh-feas}} \text{ iff } \rho \text{ is feasible in some FO-structure } \mathtt{M}$$

- $P \models \varphi$ iff $\mathsf{Exec}(P^{\neg\varphi}) \cap L_{\text{coh-feas}} = \varnothing$

Streaming Congruence Closure

- $\mathcal{A}_{\text{coh-feas}} = (Q \uplus \{q_{\text{reject}}\}, q_0, \delta)$ with $L(\mathcal{A}_{\text{coh-feas}}) = L_{\text{coh-feas}}$.
- All states in Q are accepting.
- q_{reject} is absorbing reject state, represents an infeasible execution.
- States in Q are triplets:

$$(\sim,\ d,\ F)$$

Equivalence on variables

$[x_1, x_3]$
$[x_2, x_4, x_5]$
$\vdots$

Disequalities b/w eq. classes

$[x_1, x_3] \neq [x_6]$
$\vdots$

Partial func. relationships b/w eq. classes

$f([x_1, x_3]) = [x_2, x_4, x_5]$
$\vdots$

Streaming Congruence Closure

Transitions δ update these relationships in a streaming fashion.

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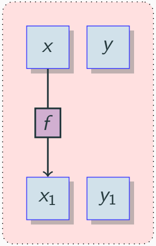
$x_1 = f(x)$



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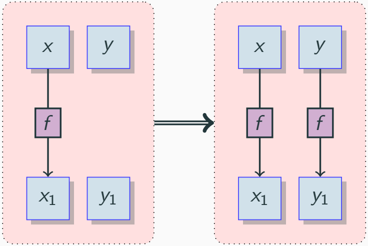
$$x_1 = f(x) \longrightarrow y_1 = f(y)$$



Streaming Congruence Closure

Transitions δ update these relationships in a streaming fashion.

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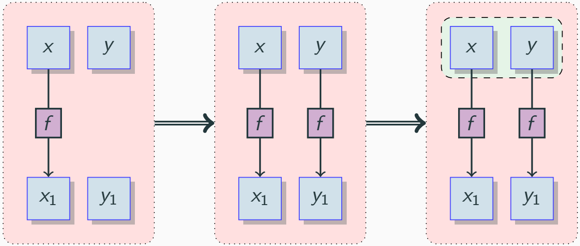
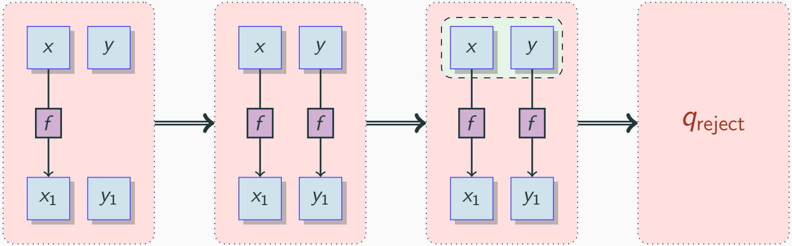


Congruence Closure

Streaming Congruence Closure

Transitions δ update these relationships in a streaming fashion.

$$x_1 = f(x) \longrightarrow y_1 = f(y) \longrightarrow \textbf{assume}(x = y) \longrightarrow \textbf{assume}(x \neq y)$$



Congruence Closure

Streaming Congruence Closure

Correctness of $\mathcal{A}_{\text{coh-feas}}$

Let $\rho \in \Pi^*$ be a coherent execution. Let $q = \delta^*(q_0, \rho)$. Then,

- If ρ is not feasible in any $\mathtt{M}$, then $q = q_{\text{reject}}$
- Otherwise, $q = (\sim, d, P)$ with
  - $\text{Term}(\rho, x) \cong_{\alpha(\rho)} \text{Term}(\rho, y)$ iff $[x]_\sim = [y]_\sim$.
  - $([x]_\sim, [y]_\sim) \in d$ iff there is $(t_x, t_y) \in \beta(\rho)$ such that $t_x \cong_{\alpha(\rho)} \text{Term}(\rho, x)$ and $t_y \cong_{\alpha(\rho)} \text{Term}(\rho, y)$.
  - $f(\text{Term}(\rho, x)) \cong_{\alpha(\rho)} \text{Term}(\rho, y)$ iff $F(f)([x]_\sim) = [y]_\sim$

Checking Coherence

Decidability of Checking Coherence [1]

There is a DFA $\mathcal{A}_{\text{check-coh}}$ such that for an execution $\rho \in \Pi^*$, we have

$$\rho \in L(\mathcal{A}_{\text{check-coh}}) \text{ iff } \rho \text{ is coherent}$$

Checking Coherence

Decidability of Checking Coherence [1]

There is a DFA $\mathcal{A}_{\text{check-coh}}$ such that for an execution $\rho \in \Pi^*$, we have

$$\rho \in L(\mathcal{A}_{\text{check-coh}}) \text{ iff } \rho \text{ is coherent}$$

- $\mathcal{A}_{\text{check-coh}}$ ignores all letters of the form "**assume**$(x \neq y)$".
- States of $\mathcal{A}_{\text{check-coh}}$ maintain $(\sim, F, B)$:
  - $\sim$ and F are as in $\mathcal{A}_{\text{coh-feas}}$
  - B keeps track of the following information: for a given equiv. class c and for a function f, if $f(c)$ has been computed before.

k-Coherence

k-Coherence

```
assume (x ≠ z);
y := n(x);
assume (y ≠ z);

y := n(y);
while (y ≠ z) {
    x := n(x);

    y := n(y);
}

φ ≡ z = n(n(x))
```

k-Coherence

```
assume (x ≠ z);
y := n(x);        ----> n(x̂)
assume (y ≠ z);

y := n(y);        ----> n(n(x̂))
while (y ≠ z) {
    x := n(x);

    y := n(y);
}

φ ≡ z = n(n(x))
```

NOT coherent

- Re-computation without storing prior equivalent terms.
- Insufficient number of program variables to store intermediate terms.

k-Coherence

```
assume (x ≠ z);
y := n(x);
assume (y ≠ z);
g := y;
y := n(y);
while (y ≠ z) {
    x := n(x);
    g := y;
    y := n(y);
}
```

$\varphi \equiv$ z = n(n(x))

1-coherent

- Can be *made coherent*.
- By adding additional ghost variables and assignments to them.
- Write-only and do not change semantics.

k-Coherence

Definition (k-Coherent Executions and Programs)

Let $k \in \mathbb{N}$. Let V be a set of variables and let $G = \{g_1, \ldots, g_k\}$ be additional ghost variables ($V \cap G = \varnothing$).
Let $\Pi_G = \Pi \cup \{\text{“}g := x\text{”} \mid g \in G, x \in V\}$.
An execution over V is k-coherent if there is an execution ρ' over Π_G such that ρ' is coherent and $\rho'\downharpoonright_{\Pi} = \rho$.
A programs is k-coherent if all its executions are.

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Theorem [1]

Checking k-coherence is decidable in PSPACE. Further, verification of k-coherent programs is decidable in PSPACE.

Verification Modulo Theories

Adding Interpretations

```
assume (T ≠ F);
if (a ≤ b) then {
   if (a ≤ c) then
      min := a;
   else min := c;
}
else {
   if (b ≤ c) then
      min := b;
   else min := c;
}
```

$\varphi \equiv \mathtt{min} \leq \mathtt{a} \wedge \mathtt{min} \leq \mathtt{b}$
$\wedge \mathtt{min} \leq \mathtt{c}$

Find the minimum of a, b and c

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Does not hold in all M.

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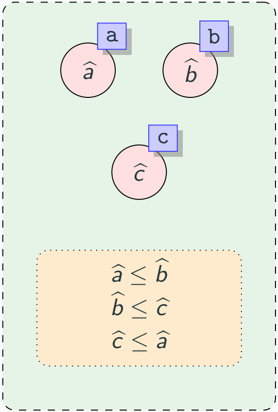

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Find the minimum of a, b and c

This program satisfies φ if $\leq$ is interpreted as a total order:

- $\forall x \cdot x \leq x$
- $\forall x, y, z \cdot x \leq y \wedge y \leq z \implies x \leq z$
- $\forall x, y \cdot x \leq y \wedge y \leq x \implies x = y$

Adding Interpretations

Definition (Verification Modulo Axioms)

Let $P \in \langle \text{stmt} \rangle$ be an uninterpreted program over vocabulary Σ. Let A be a set of first order sentences over Σ and let φ be an assertion in the following grammar.

$$\varphi ::= \texttt{true} \mid x = y \mid R(\mathbf{z}) \mid \varphi \vee \varphi \mid \neg \varphi$$

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$P \models \varphi$ modulo A iff for every execution $\rho \in \mathsf{Exec}(P)$ and for every FO structure $\mathtt{M}$ such that $\mathtt{M} \models A$ and ρ is feasible in $\mathtt{M}$, $\mathtt{M}$ satisfies $\varphi[\mathsf{val}_{\mathtt{M}}(\rho, V)/V]$.

Coherence Modulo Axioms

Coherence modulo axioms	$=$	Memoizing modulo axioms	$+$	Early Assumes modulo axioms

Example

$$A = \{\forall x, y \cdot f(x, y) = f(y, x)\}$$

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$$x_1 := f(x, y) \longrightarrow y_1 := f(y, x)$$

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re-computation modulo A

Example

$$A = \{\forall x, y \cdot f(x,y) = f(y,x)\}$$

re-computation modulo A

$$x_1 := f(x,y) \longrightarrow y_1 := f(y,x)$$

$$x_1 := f(x,y) \rightarrow y_1 := f(y,x') \longrightarrow z := g(x_1) \longrightarrow z' := g(y_1) \rightarrow \textbf{assume}(x = x')$$

Example

$A = \{\forall x, y \cdot f(x, y) = f(y, x)\}$

$x_1 := f(x, y) \longrightarrow y_1 := f(y, x)$

re-computation
modulo A

$x_1 := f(x, y) \rightarrow y_1 := f(y, x') \longrightarrow z := g(x_1) \longrightarrow z' := g(y_1) \rightarrow \textbf{assume}(x = x')$

Implied equality
$z = z'$

Memoizing Modulo Axioms

Definition (Memoizing modulo axioms)

Let A be a set of axioms and let $\rho \in \Pi^*$ be an execution. Then, ρ is said to be memoizing modulo A if the following holds.

Let $\sigma' = \sigma \cdot$ "$x := f(\mathbf{z})$" be a prefix of ρ. If there is a term $t' \in \text{ComputedTerms}(\sigma)$ such that $t' \cong_{A \cup \kappa(\sigma)} \text{Term}(\sigma', x)$, then there must exist some variable $y \in V$ such that $\text{Term}(\sigma, y) \cong_{A \cup \kappa(\sigma)} t$.

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Definition (Memoizing modulo axioms)

Let A be a set of axioms and let $\rho \in \Pi^*$ be an execution. Then, ρ is said to be memoizing modulo A if the following holds.

Let $\sigma' = \sigma \cdot$ "$x := f(\mathbf{z})$" be a prefix of ρ. If there is a term $t' \in \text{ComputedTerms}(\sigma)$ such that $t' \cong_{A \cup \kappa(\sigma)} \text{Term}(\sigma', x)$, then there must exist some variable $y \in V$ such that $\text{Term}(\sigma, y) \cong_{A \cup \kappa(\sigma)} t$.

Here,

$$\begin{aligned}
\kappa(\varepsilon) &= \varnothing \\
\kappa(\rho \cdot \text{“}\textbf{assume}(x = y)\text{”}) &= \kappa(\rho) \cup \{(\text{Term}(\rho, x) = \text{Term}(\rho, y))\} \\
\kappa(\rho \cdot \text{“}\textbf{assume}(x \neq y)\text{”}) &= \kappa(\rho) \cup \{(\text{Term}(\rho, x) \neq \text{Term}(\rho, y))\} \\
\kappa(\rho \cdot \text{“}R(z_1, \ldots)\text{”}) &= \kappa(\rho) \cup \{R(\text{Term}(\rho, z_1), \ldots)\} \\
\kappa(\rho \cdot a) &= \kappa(\rho) \quad \text{otherwise}
\end{aligned}$$

Early Assumes Modulo Axioms

Definition (Early assumes modulo axioms)

Let A be a set of axioms and let $\rho \in \Pi^*$ be an execution. Then, ρ is said to have early assumes modulo A if the following holds.

Let $\sigma' = \sigma \cdot$ "**assume**(c)" be a prefix of ρ, where c is any of $x = y$, $x \neq y$, $R(\mathbf{z})$, or $\neg R(\mathbf{z})$.
Let $t \in \mathrm{ComputedTerms}(\sigma)$ be a term computed in σ such that t has been *dropped*, i.e., for every $x \in V$, we have $\mathrm{Term}(\sigma, x) \ncong_{A \cup \kappa(\sigma)} t$.
For any term $t' \in \mathrm{ComputedTerms}(\sigma)$, if $t \cong_{A \cup \kappa(\sigma')} t'$, then $t \cong_{A \cup \kappa(\sigma)} t'$.

Verification Modulo Axioms - Decidability Landscape [2]

Relational axioms	Decidability
EPR	✗
Reflexivity	✓
Irreflexivity	✓
Symmetry	✓
Transitivity	✓
Partial Order	✓
Total Order	✓

Functional axioms	Decidability
Associativity	✗
Commutativity	✓
Idempotence	✓

Combinations	Decidability
All combinations of decidable axioms	✓

Thank You!

Coherence Modulo Commutativity

Homomorphism h^f_{comm} uses auxiliary variable $v^* \notin V$:

$$h^f_{\text{comm}}(a) = \begin{cases} a \cdot \text{“}v^* := f(y,x)\text{”} \cdot \text{“}\textbf{assume}(z = v^*)\text{”} & \text{if } a = \text{“}z := f(x,y)\text{”} \\ a & \text{otherwise} \end{cases}$$

Coherence Modulo Commutativity

An execution ρ is coherent modulo A iff $h^f_{\text{comm}}(a)$ is coherent modulo $\varnothing$.

Feasibility Modulo Commutativity

An execution ρ is feasible modulo A iff $h^f_{\text{comm}}(a)$ is feasible modulo $\varnothing$.

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