SUBATOMIC PROOF SYSTEMS AND DECISION TREES

Alessio Guglielmi
joint work with Chris Barrett and Victoria Barrett

OWLS 5/5/21

Talk available from AG's home page and at https://people.bath.ac.uk/ag248/t/SPSDT.pdf

All about deep inference at http://alessio.guglielmi.name/res/cos

Last Slide: Subatomic Proof System for Classical Propositional Logic

$$\frac{(A \,\beta\, B)\ \alpha\ (C \,\beta^{\bullet}\, D)}{(A \,\alpha\, C)\ \beta\ (B \,\alpha^{\bullet}\, D)}$$

$$\check{\vee} = \check{\wedge} = \vee$$

$$\hat{\vee} = \hat{\wedge} = \wedge$$

$$\check{\mathbf{a}} = \hat{\mathbf{a}} = \mathbf{a}$$

$$\alpha \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \ldots\}$$

Last Slide: Subatomic Proof System for Classical Propositional Logic

$$\frac{(A \,\beta\, B)\ \alpha\ (C\ \beta'\, D)}{(A \,\alpha\, C)\ \beta\ (B\ \alpha'\, D)}$$

$\alpha \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \ldots\}$

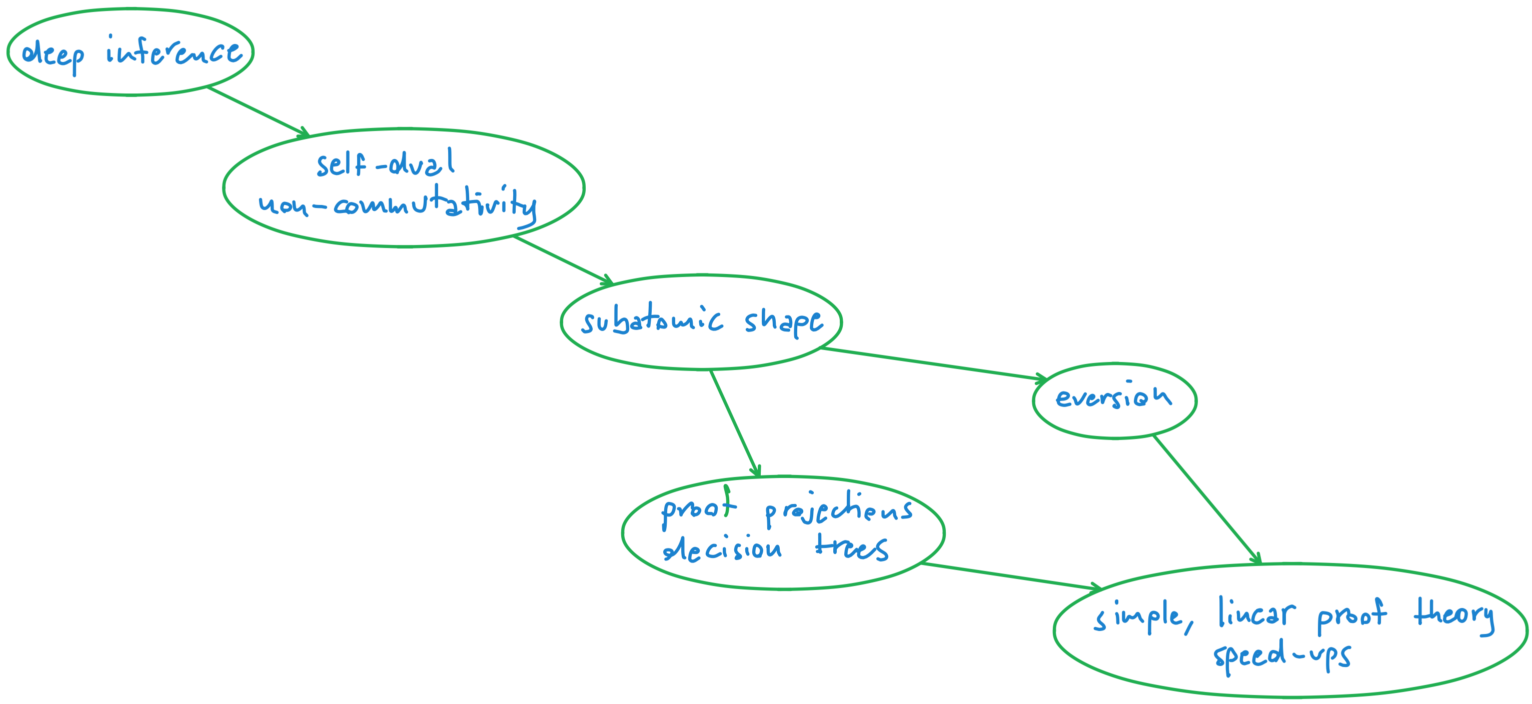
$\check{\vee} = \check{\wedge} = \vee$

$\hat{\vee} = \hat{\wedge} = \wedge$

$\check{\mathbf{a}} = \hat{\mathbf{a}} = \mathbf{a}$

- all the structure needed for normalisation is here
- linear (totally!)
- normalisation at the standard level can be recovered from the subatomic one

PLAN



deep inference
self-dual non-commutativity
subatomic shape
eversion
proof projections decision trees
simple, linear proof theory speed-ups

Deep Inference — Speed-Ups

drinker formula $\exists x . \forall y . (\overline{Px} \vee Py)$

Deep Inference — Speed-Ups

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash Pz,\ \overline{Pz}}{\vdash Pz,\ \overline{Pz},\ Py}}{\vdash \overline{Px},\ Pz,\ \overline{Pz},\ Py}}{\vdash \overline{Px},\ Pz,\ \overline{Pz} \vee Py}}{\vdash \overline{Px} \vee Pz,\ \overline{Pz} \vee Py}}{\vdash \overline{Px} \vee Pz,\ \forall y.(\overline{Pz} \vee Py)}}{\vdash \overline{Px} \vee Pz,\ \exists x.\forall y.(\overline{Px} \vee Py)}}{\vdash \forall y.(\overline{Px} \vee Py),\ \exists x.\forall y.(\overline{Px} \vee Py)}}{\vdash \exists x.\forall y.(\overline{Px} \vee Py),\ \exists x.\forall y.(\overline{Px} \vee Py)}}{\vdash \exists x.\forall y.(\overline{Px} \vee Py)}$$

proof in the sequent calculus

Deep Inference — Speed-Ups

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash Pz,\ \overline{P}z}{\vdash Pz,\ \overline{P}z,\ Py}}{\vdash \overline{P}x,\ Pz,\ \overline{P}z,\ Py}}{\vdash \overline{P}x,\ Pz,\ \overline{P}z \vee Py}}{\vdash \overline{P}x \vee Pz,\ \overline{P}z \vee Py}}{\vdash \overline{P}x \vee Pz,\ \forall y.(\overline{P}z \vee Py)}}{\vdash \overline{P}x \vee Pz,\ \exists x.\forall y.(\overline{P}x \vee Py)}}{\vdash \forall y.(\overline{P}x \vee Py),\ \exists x.\forall y.(\overline{P}x \vee Py)}}{\vdash \exists x.\forall y.(\overline{P}x \vee Py),\ \exists x.\forall y.(\overline{P}x \vee Py)}}{\vdash \exists x.\forall y.(\overline{P}x \vee Py)}$$

proof in the sequent calculus

bureaucracy requires a contraction

Deep Inference — Speed-Ups

$$
\begin{array}{c}
\vdash Pz,\ \overline{Pz} \\ \hline
\vdash Pz,\ \overline{Pz},\ Py \\ \hline
\vdash \overline{Px},\ Pz,\ \overline{Pz},\ Py \\ \hline
\vdash \overline{Px},\ Pz,\ \overline{Pz} \vee Py \\ \hline
\vdash \overline{Px} \vee Pz,\ \overline{Pz} \vee Py \\ \hline
\vdash \overline{Px} \vee Pz,\ \forall y.(\overline{Pz} \vee Py) \\ \hline
\vdash \overline{Px} \vee Pz,\ \exists x.\forall y.(\overline{Px} \vee Py) \\ \hline
\vdash \forall y.(\overline{Px} \vee Py),\ \exists x.\forall y.(\overline{Px} \vee Py) \\ \hline
\vdash \exists x.\forall y.(\overline{Px} \vee Py),\ \exists x.\forall y.(\overline{Px} \vee Py) \\ \hline
\vdash \exists x.\forall y.(\overline{Px} \vee Py)
\end{array}
$$

proof in the sequent calculus

bureaucracy requires a contraction

$$
\begin{array}{c}
t \\ \hline
\exists x.\overline{Px} \vee \forall y.Py \\ \hline
\exists x.\boxed{\dfrac{\overline{Px} \vee \forall y.Py}{\forall y.(\overline{Px} \vee Py)}}
\end{array}
$$

proof in deep inference

Deep Inference — Speed-Ups

$$
\begin{array}{c}
\vdash Pz, \overline{Pz} \\ \hline
\vdash Pz, \overline{Pz}, Py \\ \hline
\vdash \overline{Px}, Pz, \overline{Pz}, Py \\ \hline
\vdash \overline{Px}, Pz, \overline{Pz} \vee Py \\ \hline
\vdash \overline{Px} \vee Pz, \overline{Pz} \vee Py \\ \hline
\vdash \overline{Px} \vee Pz, \forall y.(\overline{Pz} \vee Py) \\ \hline
\vdash \overline{Px} \vee Pz, \exists x.\forall y.(\overline{Px} \vee Py) \\ \hline
\vdash \forall y.(\overline{Px} \vee Py), \exists x.\forall y.(\overline{Px} \vee Py) \\ \hline
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proof in the sequent calculus

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\exists x.\boxed{\begin{array}{c} \overline{Px} \vee \forall y.Py \\ \hline \forall y.(\overline{Px} \vee Py) \end{array}}
\end{array}
$$

drinker formula

proof in deep inference

Deep Inference — Speed-Ups

$$
\begin{array}{c}
\vdash Pz,\ \overline{Pz} \\ \hline
\vdash Pz,\ \overline{Pz},\ Py \\ \hline
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\vdash \exists x.\forall y.(\overline{Px} \vee Py)
\end{array}
$$

sequent calculus

bureaucracy requires a contraction

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\begin{array}{c}
t \\ \hline
\exists x.\overline{Px} \vee \forall y.Py \\ \hline
\exists x.\boxed{\begin{array}{c} \overline{Px} \vee \forall y.Py \\ \hline \forall y.(\overline{Px} \vee Py) \end{array}}
\end{array}
$$

deep inference = inferences inside formulae

Deep Inference – Speed-Ups

$$
\begin{array}{r}
\vdash Pz,\ \overline{Pz} \\ \hline
\vdash Pz,\ \overline{Pz},\ Py \\ \hline
\vdash \overline{Px},\ Pz,\ \overline{Pz},\ Py \\ \hline
\vdash \overline{Px},\ Pz,\ \overline{Pz} \vee Py \\ \hline
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sequent calculus

bureaucracy requires a contraction

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\end{array}
$$

deep inference

=

inferences inside formulae

deep inference does not require a contraction

Deep Inference — Speed-Ups

Corollary of [Aguilera-Baaz, JSL 2019]

Deep inference has a nonelementary speed-up over cut-free Gentzen proofs of the predicate calculus.

$$\cfrac{\cfrac{t}{\exists x.\overline{Px} \vee \forall y.Py}}{\exists x.\boxed{\cfrac{\overline{Px} \vee \forall y.Py}{\forall y.(\overline{Px} \vee Py)}}}$$

deep inference

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inferences inside formulae

deep inference does not require a contraction

Deep Inference — Speed-Ups

Corollary of [Aguilera-Baaz, JSL 2019]

Deep inference has a nonelementary speed-up over cut-free Gentzen proofs of the predicate calculus.

Theorems

The speed-up for cut-free propositional proofs is exponential.
[Bruscoli, Guglielmi, ACM ToCL 2009]

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deep inference
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inferences inside formulae

deep inference does not require a contraction

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Theorems

The speed-up for cut-free propositional proofs is exponential.
[Bruscoli, Guglielmi, ACM ToCL 2009]

Cut-elimination for propositional classical logic is quasipolynomial.
[Jeřábek, JLC 2009]

$$\frac{\dfrac{t}{\exists x.\overline{Px} \vee \forall y.Py}}{\exists x.\boxed{\dfrac{\overline{Px} \vee \forall y.Py}{\forall y.(\overline{Px} \vee Py)}}}$$

deep inference
=
inferences inside formulae

deep inference does not require a contraction

Deep Inference — Expressiveness

proof system

identity rule

$$\mathsf{id}\,\frac{}{a \parr \bar{a}}$$

dual atoms in a par

structure related to some formula/proof

a

$\bar{a}$

proof of the formula in the proof system

$$\frac{}{a \parr \bar{a}}$$

Deep Inference — Expressiveness

proof system

MLL

identity rule

$$\mathrm{id}\ \frac{}{a \parr \bar{a}} \qquad \parr\!\check{\otimes}\ \frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)}$$

par/tensor rule

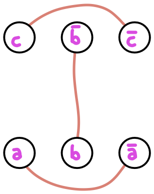
$\parr$, $\otimes$ ass., comm.

structure

par/tensor commutative

e.g.:

- space
- parallel processes
- LES conflict
- ...



proof

e.g., three synchronisation events:

$$\frac{}{a \parr \bar{a}} \quad \frac{}{b \parr \bar{b}} \quad \frac{}{c \parr \bar{c}}$$

DEEP INFERENCE — EXPRESSIVENESS

proof system

MLL

identity rule

$$\text{id}\ \frac{}{a \parr \bar{a}}$$

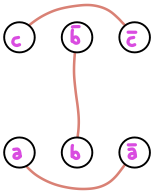
$$\frac{\vdash A, B \qquad \vdash C, D}{\vdash A \otimes C, B, D}$$

par / tensor rule

$\parr$, $\otimes$ ass., comm.

cfr. sequent calculus
$\otimes$ = branching

structure $\longleftrightarrow$



par/tensor
commutative

e.g.:
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proof

e.g., three synchronisation events:

$$\frac{}{a \parr \bar{a}} \quad \frac{}{b \parr \bar{b}} \quad \frac{}{c \parr \bar{c}}$$

Deep Inference — Expressiveness

proof system

MLL
+ seq

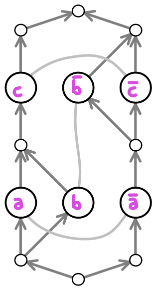
$$\mathsf{id}\,\frac{}{a \parr \bar{a}} \qquad \parr\otimes\,\frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\lhd\,\frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

$\parr$, $\otimes$ ass., comm.

$\lhd$ ass., non-comm., self-dual, i.e. $\overline{A \lhd B} = \bar{A} \lhd \bar{B}$

structure

par/tensor
commutative



seq
non-commutative
self-dual

e.g.:
- time
- sequential processes
- LES causality
- ...

proof

$$\frac{}{a \parr \bar{a}} \qquad \frac{}{b \parr \bar{b}} \qquad \frac{}{c \parr \bar{c}}$$

$$\boxed{\boxed{a \parr b} \lhd c} \parr \boxed{\bar{a} \lhd \boxed{\bar{b} \parr \bar{c}}}$$

Deep Inference — Expressiveness

proof system

MLL
+ seq

$$\mathsf{id}\,\frac{}{a \parr \bar{a}} \qquad \parr\check{\otimes}\,\frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\triangleleft\,\frac{(A \parr B) \triangleleft (C \parr D)}{(A \triangleleft C) \parr (B \triangleleft D)}$$

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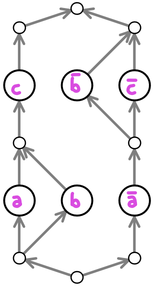
structure

par/tensor
commutative



seq
non-commutative
self-dual

e.g.:
- time
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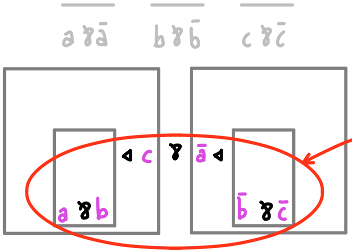
proof



conclusion

$((a \parr b) \triangleleft c) \parr (\bar{a} \triangleleft (\bar{b} \parr \bar{c}))$

Deep Inference – Expressiveness

proof system

MLL + seq

$$\mathsf{id}\ \frac{}{a \parr \bar{a}} \qquad \parr\check{\otimes}\ \frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\lhd\ \frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

$\parr$, $\otimes$ ass., comm.

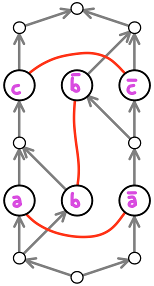
$\lhd$ ass., non-comm., self-dual, i.e. $\overline{A \lhd B} = \bar{A} \lhd \bar{B}$

how do we prove it?

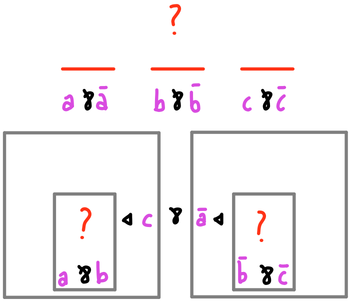
structure

par/tensor commutative

seq non-commutative self-dual



proof



Deep Inference — Expressiveness

proof theory: [Guglielmi, ACM ToCL 2007]

semantics: [Blute, Panangaden, Slavnov, Applied Categorical Structures 2012]

proof system

BV = MLL + seq + mix + mix0

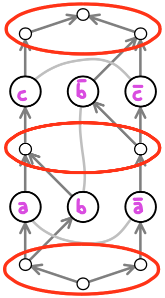
$$\mathsf{id}\ \frac{\circ}{a \parr \bar{a}} \qquad \parr\otimes\ \frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\triangleleft\ \frac{(A \parr B) \triangleleft (C \parr D)}{(A \triangleleft C) \parr (B \triangleleft D)}$$

$\parr$, $\otimes$ ass., comm. $\qquad \circ$ unit for $\parr, \otimes, \triangleleft$

$\triangleleft$ ass., non-comm., self-dual, i.e. $\overline{A \triangleleft B} = \bar{A} \triangleleft \bar{B}$

structure

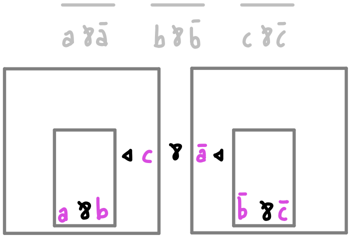
par/tensor commutative



units

seq non-commutative self-dual

proof



Deep Inference — Expressiveness

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proof system

BV

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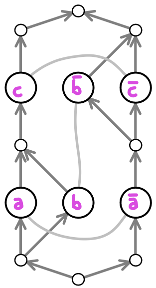
$\triangleleft$ ass., non-comm., self-dual, i.e. $\overline{A \triangleleft B} = \bar{A} \triangleleft \bar{B}$

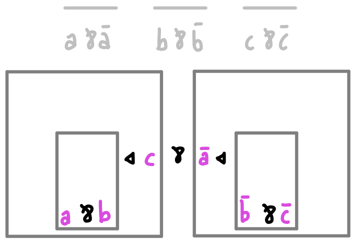
structure

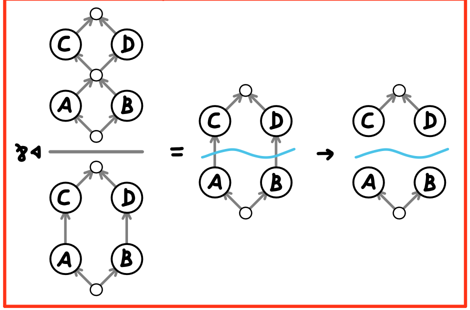


par/tensor commutative

seq non-commutative self-dual

proof





DEEP INFERENCE — EXPRESSIVENESS

proof system

BV

$$\mathsf{id}\,\frac{\circ}{a \parr \bar{a}} \qquad \parr\otimes\,\frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\triangleleft\,\frac{(A \parr \circ) \triangleleft (\circ \parr D)}{(A \triangleleft \circ) \parr (\circ \triangleleft D)}$$

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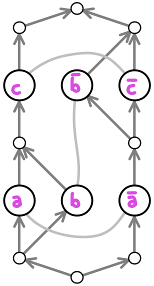
structure

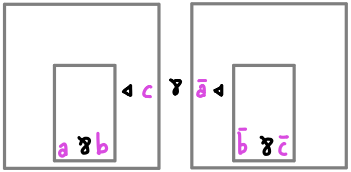
par/tensor commutative

seq non-commutative self-dual

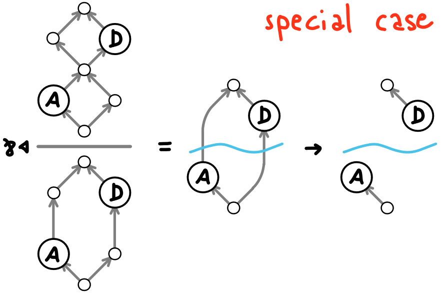


proof

$$\frac{}{a \parr \bar{a}} \qquad \frac{}{b \parr \bar{b}} \qquad \frac{}{c \parr \bar{c}}$$



special case



Deep Inference — Expressiveness

proof system

BV

$$\mathsf{id}\,\frac{\circ}{a \parr \bar a}\qquad \parr\check{\otimes}\,\frac{(A\parr B)\otimes(C\parr D)}{(A\otimes C)\parr(B\parr D)}\qquad \parr\triangleleft\,\frac{(A\parr\circ)\triangleleft(\circ\parr D)}{(A\triangleleft\circ)\parr(\circ\triangleleft D)}$$

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structure

par/tensor commutative

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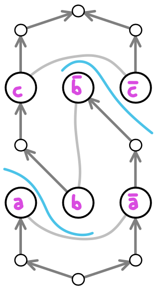


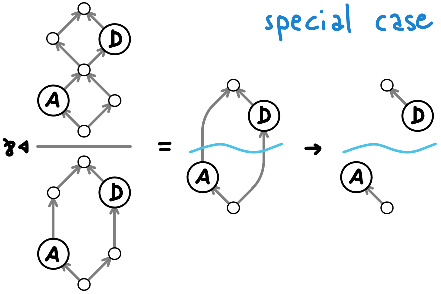
two steps of $\parr\triangleleft$

proof

$$\frac{}{a\parr\bar a}\qquad \frac{}{b\parr\bar b}\qquad \frac{}{c\parr\bar c}$$

$$\left(\frac{a\triangleleft b}{a\parr b}\triangleleft c\right)\parr\left(\bar a\triangleleft\frac{\bar b\triangleleft\bar c}{\bar b\parr\bar c}\right)$$

special case



Deep Inference — Expressiveness

BV

proof system

$$\mathsf{id}\,\frac{\circ}{a \parr \bar{a}} \qquad \parr\!\otimes\,\frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\!\lhd\,\frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

$\parr, \otimes$ ass., comm. $\qquad$ $\circ$ unit for $\parr, \otimes, \lhd$

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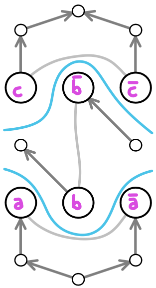
structure

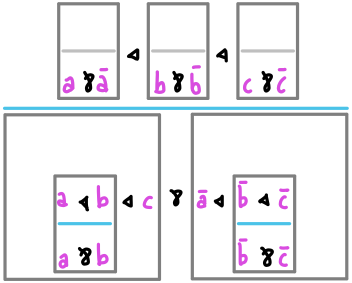
par/tensor commutative

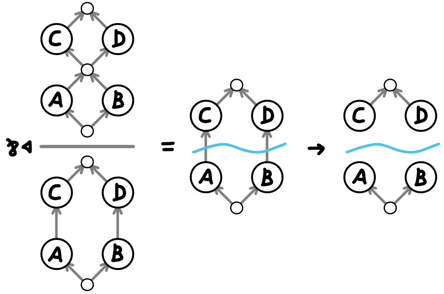
seq non-commutative self-dual



two steps of $\parr\!\lhd$

proof





Deep Inference — Expressiveness

proof system

BV

$$\mathsf{id}\,\frac{\circ}{a \parr \bar{a}} \qquad \parr\check{\otimes}\,\frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\triangleleft\,\frac{(A \parr B) \triangleleft (C \parr D)}{(A \triangleleft C) \parr (B \triangleleft D)}$$

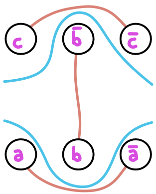
$\parr$, $\otimes$ ass., comm. $\qquad$ $\circ$ unit for $\parr, \otimes, \triangleleft$

$\triangleleft$ ass., non-comm., self-dual, i.e. $\overline{A \triangleleft B} = \bar{A} \triangleleft \bar{B}$

structure

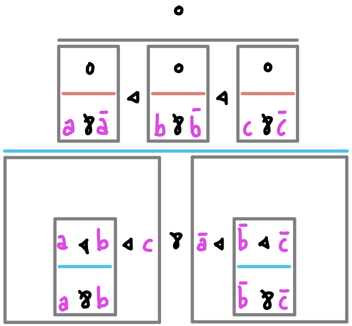
par/tensor
commutative

seq
non-commutative
self-dual



three steps of id

proof



Deep Inference — Expressiveness

proof system

BV

$$\mathsf{id}\ \frac{\circ}{a \parr \bar{a}} \qquad \parr\otimes\ \frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\triangleleft\ \frac{(A \parr B) \triangleleft (C \parr D)}{(A \triangleleft C) \parr (B \triangleleft D)}$$

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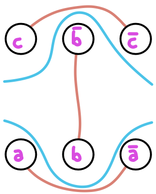
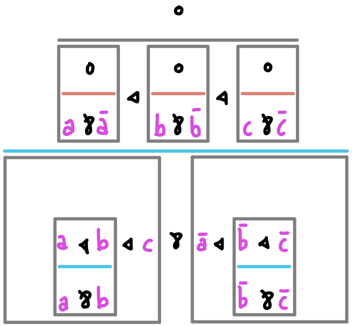
structure

par/tensor
commutative

seq
non-commutative
self-dual



proof



one cannot do this in Gentzen's theory

Deep Inference – Expressiveness

proof system

BV

$$\text{id}\ \frac{\circ}{a \parr \bar{a}} \qquad \parr\otimes\ \frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\lhd\ \frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

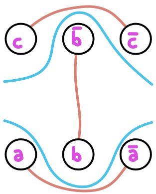
$\parr$, $\otimes$ ass., comm. $\qquad$ $\circ$ unit for $\parr, \otimes, \lhd$

$\lhd$ ass., non-comm., self-dual, i.e. $\overline{A \lhd B} = \bar{A} \lhd \bar{B}$

structure

par/tensor
commutative

seq
non-commutative
self-dual



proof

$$\dfrac{\circ}{\dfrac{\dfrac{\circ}{a \parr \bar{a}} \lhd \dfrac{\circ}{b \parr \bar{b}} \lhd \dfrac{\circ}{c \parr \bar{c}}}{\left(\dfrac{a \lhd b}{a \parr b} \lhd c\right) \parr \left(\bar{a} \lhd \dfrac{\bar{b} \lhd \bar{c}}{\bar{b} \parr \bar{c}}\right)}}$$

one cannot do this in Gentzen's theory because $\lhd$ branching is different from (standard) $\otimes$ branching

Deep Inference — Expressiveness

proof system

BV

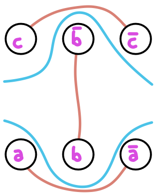
$$\mathsf{id}\,\frac{\circ}{a \parr \bar{a}} \qquad \parr\otimes\,\frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\lhd\,\frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

$\parr, \otimes$ ass., comm. $\qquad \circ$ unit for $\parr, \otimes, \lhd$

$\lhd$ ass., non-comm., self-dual, i.e. $\overline{A \lhd B} = \bar{A} \lhd \bar{B}$

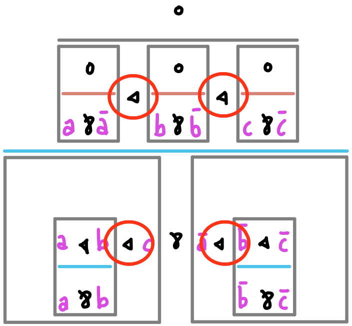
structure

par/tensor
commutative



seq
non-commutative
self-dual

proof



deep inference
=
composing derivations with any connective

Deep Inference – Expressiveness

proof system

BV

$$\text{id}\ \frac{\circ}{a \parr \bar{a}} \qquad \parr\otimes\ \frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\lhd\ \frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

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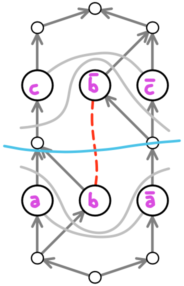
structure



any other step would break some identity, e.g.:

$$\frac{(a \parr b \parr \bar{a}) \lhd (c \parr \bar{b} \parr \bar{c})}{((a \parr b) \lhd c) \parr (\bar{a} \lhd (\bar{b} \parr \bar{c}))}$$

proof

$$\frac{\circ}{\left(\frac{\circ}{a \parr \bar{a}}\right) \lhd \left(\frac{\circ}{b \parr \bar{b}}\right) \lhd \left(\frac{\circ}{c \parr \bar{c}}\right)}$$

$$\frac{}{\left(\left(\frac{a \lhd b}{a \parr b}\right) \lhd c\right) \parr \left(\bar{a} \lhd \left(\frac{\bar{b} \lhd \bar{c}}{\bar{b} \parr \bar{c}}\right)\right)}$$

Deep Inference — Expressiveness

proof system

BV

$$\mathsf{id}\ \frac{\circ}{a \parr \bar{a}} \qquad \parr\otimes\ \frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\triangleleft\ \frac{(A \parr B) \triangleleft (C \parr D)}{(A \triangleleft C) \parr (B \triangleleft D)}$$

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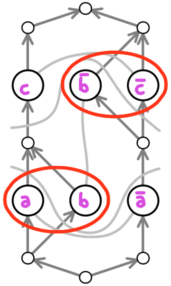
$\triangleleft$ ass., non-comm., self-dual, i.e. $\overline{A \triangleleft B} = \bar{A} \triangleleft \bar{B}$

structure

'locks'



any other step would break some identity

proof

$$\cfrac{\circ}{\cfrac{\dfrac{\circ}{a \parr \bar{a}} \triangleleft \dfrac{\circ}{b \parr \bar{b}} \triangleleft \dfrac{\circ}{c \parr \bar{c}}}{\left(\dfrac{a \triangleleft b}{a \parr b} \triangleleft c\right) \parr \left(\bar{a} \triangleleft \dfrac{\bar{b} \triangleleft \bar{c}}{\bar{b} \parr \bar{c}}\right)}}$$

'unlocking the locks' at alternation-depth 1

Deep Inference — Expressiveness

proof system

BV

$$\mathsf{id}\ \frac{\circ}{a \parr \bar{a}} \qquad \parr\check{\otimes}\ \frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\lhd\ \frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

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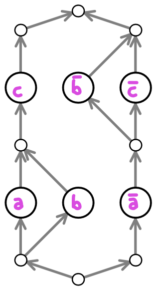
structure S_1



proof

$$\frac{\circ}{\dfrac{\left(\dfrac{\circ}{a \parr \bar{a}}\right) \lhd \left(\dfrac{\circ}{b \parr \bar{b}}\right) \lhd \left(\dfrac{\circ}{c \parr \bar{c}}\right)}{\left(\dfrac{a \lhd b}{a \parr b}\right) \lhd c \parr \bar{a} \lhd \left(\dfrac{\bar{b} \lhd \bar{c}}{\bar{b} \parr \bar{c}}\right)}}$$

'unlocking the locks' at alternation-depth 1

Deep Inference — Expressiveness

proof system

BV

$$\mathsf{id}\ \frac{\circ}{a \parr \bar{a}} \qquad \parr\!\otimes\ \frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\!\lhd\ \frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

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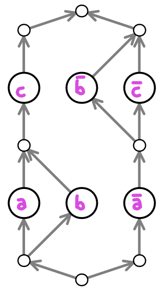
structure S_1



proof

$$\dfrac{\dfrac{\circ}{\dfrac{\circ}{a \parr \bar a} \lhd \dfrac{\circ}{b \parr \bar b} \lhd \dfrac{\circ}{c \parr \bar c}}}{\left(\dfrac{a \lhd b}{a \parr b} \lhd c\right) \parr \left(\bar a \lhd \dfrac{\bar b \lhd \bar c}{\bar b \parr \bar c}\right)}$$

'unlocking the locks' at alternation-depth 1

Deep Inference — Expressiveness

proof system

BV

$$\mathsf{id}\ \frac{\circ}{a \parr \bar{a}} \qquad \parr\otimes\ \frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\lhd\ \frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

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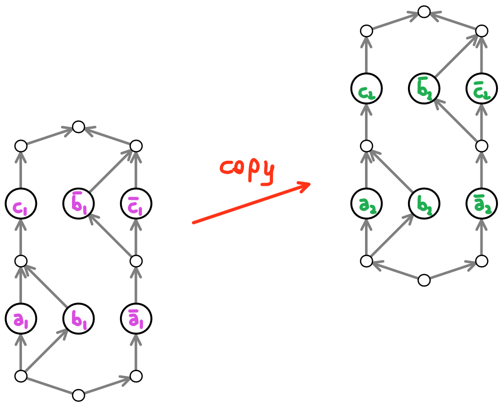
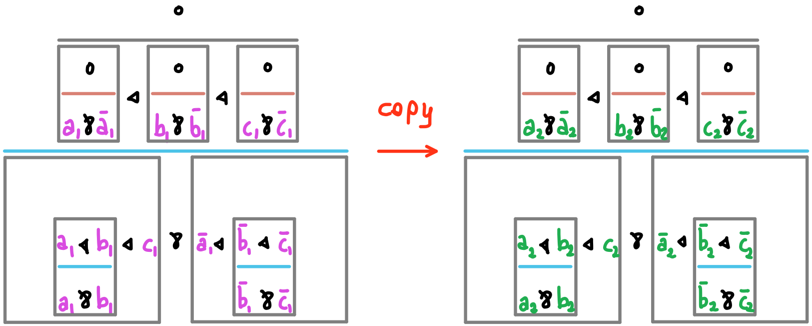
structure S_2



proof



Deep Inference — Expressiveness

proof system

BV

$$\mathsf{id}\,\frac{\circ}{a \parr \bar{a}} \qquad \parr\check{\otimes}\,\frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\lhd\,\frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

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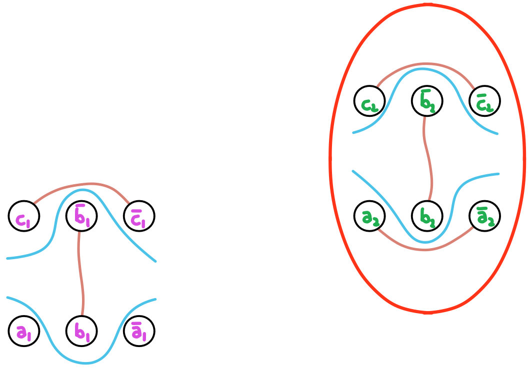
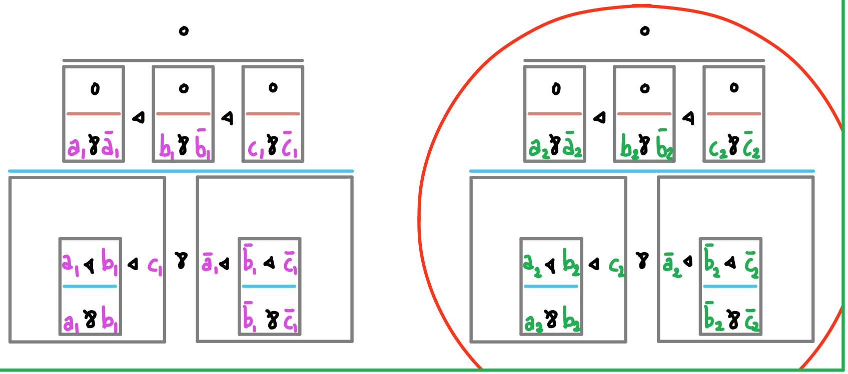


proof



Deep Inference — Expressiveness

proof system

BV

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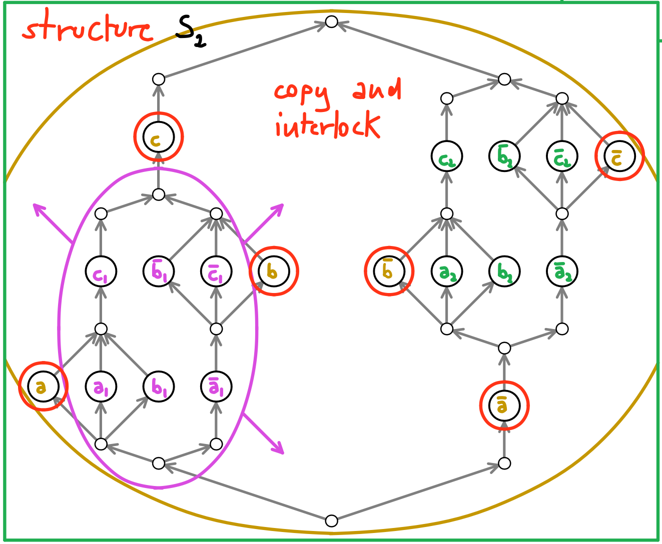

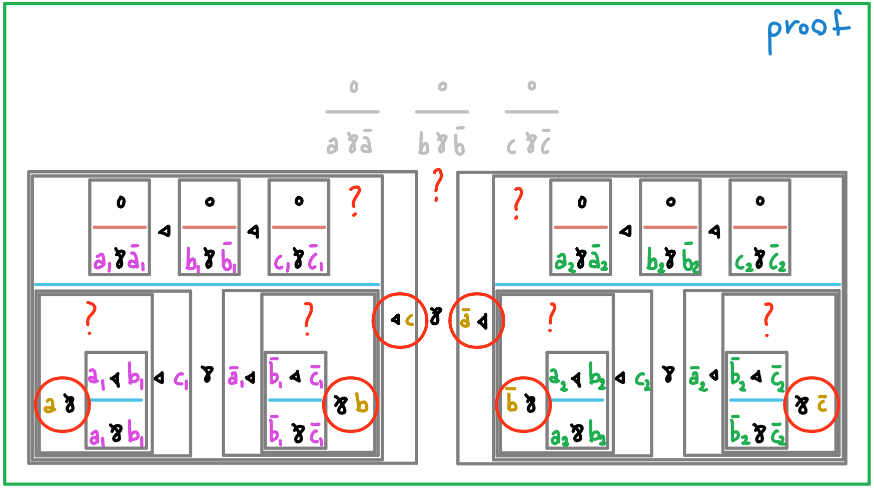

Deep Inference — Expressiveness

proof system

BV

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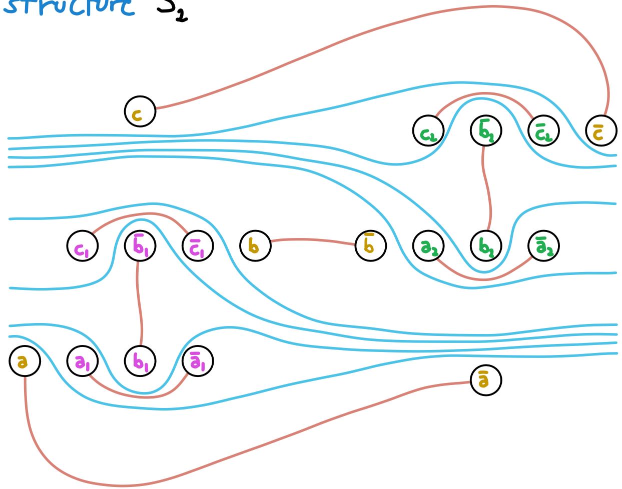
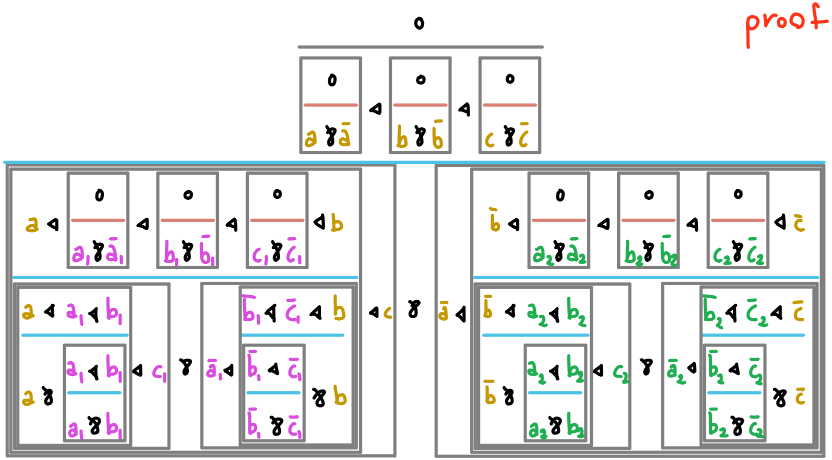
structure S_2



proof



Deep Inference — Expressiveness

proof system

BV

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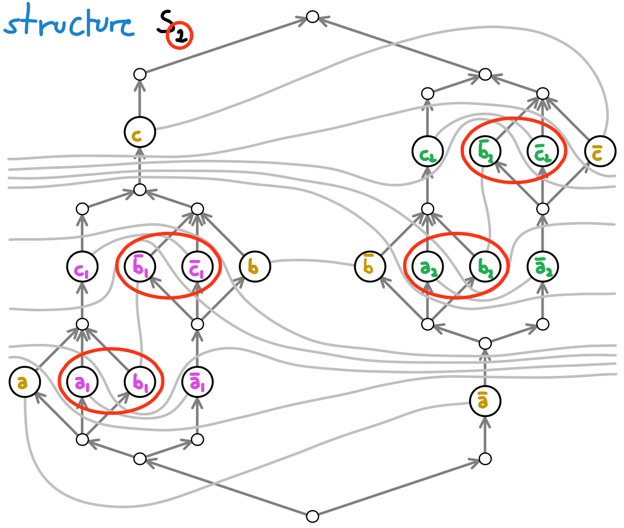
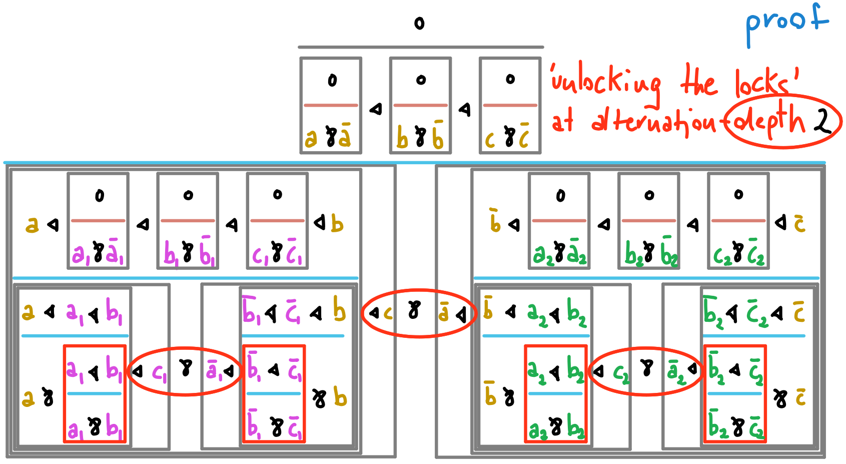
structure S_2



proof



Deep Inference — Expressiveness

Repeat the construction:

$S_1, S_2, \ldots, S_n, \ldots$

proof system

BV

$$\mathsf{id}\,\frac{\circ}{a \parr \bar{a}} \qquad \parr\check{\otimes}\,\frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\triangleleft\,\frac{(A \parr B) \triangleleft (C \parr D)}{(A \triangleleft C) \parr (B \triangleleft D)}$$

$\parr, \otimes$ ass., comm. $\quad \circ$ unit for $\parr, \otimes, \triangleleft$

$\triangleleft$ ass., non-comm., self-dual, i.e. $\overline{A \triangleleft B} = \bar{A} \triangleleft \bar{B}$

structure S_n

Theorem [Tiu, LMCS 2006] BV can only have a linear cut-free proof system in deep inference.

Proof Given any shallow-inference system whose maximum depth is m, take $n > m$. S_n is not provable in that system because the locks are unreachable.

proof

'unlocking the locks' at alternation depth n

Deep Inference — Expressiveness

BV

$$\mathsf{id}\,\frac{\circ}{a \parr \bar{a}} \qquad \parr\!\check{\otimes}\,\frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\!\lhd\,\frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

$\parr$, $\otimes$ ass., comm. $\qquad$ $\circ$ unit for $\parr$, $\otimes$, $\lhd$

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Theorem [Tiu, LMCS 2006] BV can only have a linear cut-free proof system in deep inference.

Proof Given any shallow-inference system whose maximum depth is m, take $n > m$. S_n is not provable in that system because the locks are unreachable.

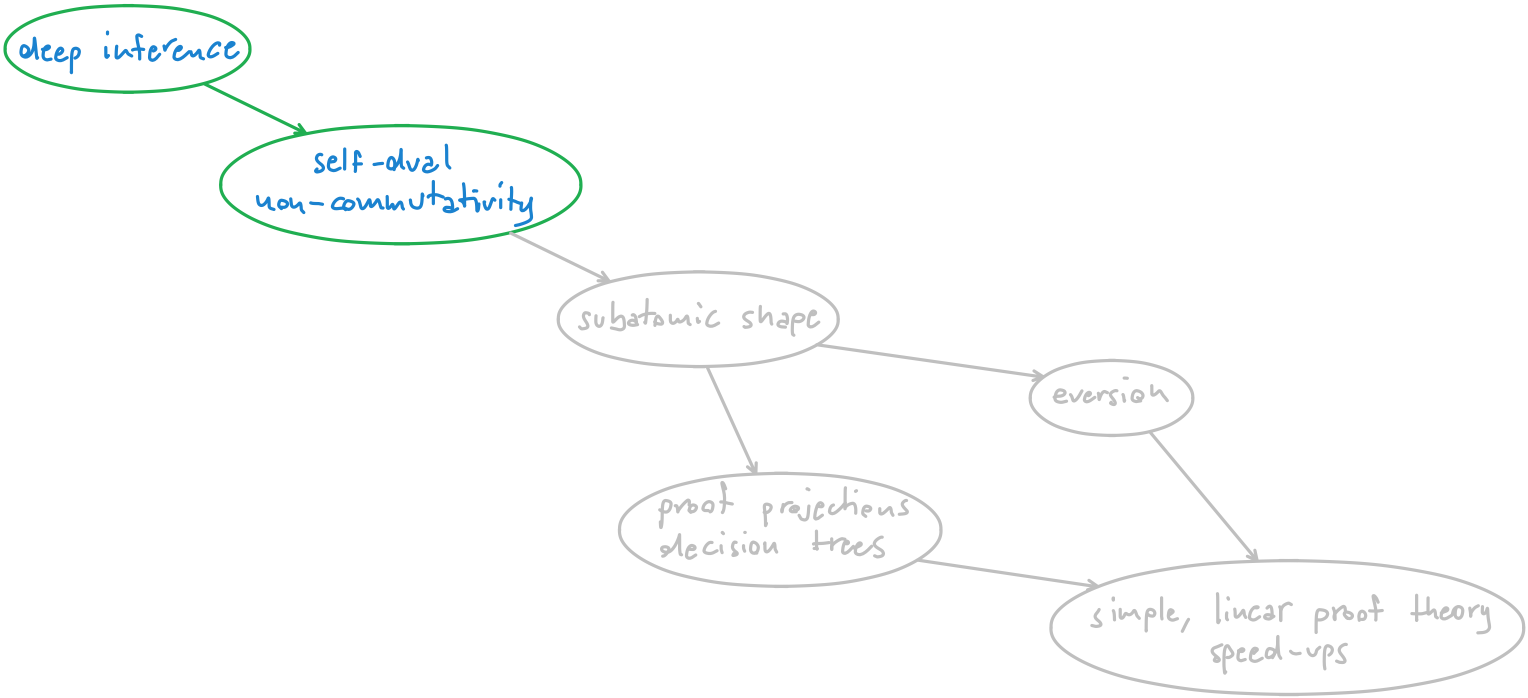
developments

extensions and theory: Aler Tubella, Blute, Guglielmi, Kahramanoğulları, Panangaden, Slavnov, Straßburger

computational models: Aman, Bruscoli, Ciobanu, Horne, Mauw, Roversi, Tiu

quantum theory: Blute, Guglielmi, Ivanov, Panangaden, Straßburger

PLAN


deep inference
self-dual
non-commutativity
subatomic shape
eversion
proof projections
decision trees
simple, linear proof theory
speed-ups

Cut Elimination via Subatomic Projections in Propositional Logic with Decision Trees

Theorem Given a proof of A, we can build a cut-free proof of A.

$$\mathord{\parr}\mathord{\otimes}\ \frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \mathord{\parr}\mathord{\lhd}\ \frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

$\parr$, $\otimes$ ass., comm.

$\lhd$ ass., non-comm., self-dual

Cut Elimination via Subatomic Projections in Propositional Logic with Decision Trees

Theorem Given a proof of A, we can build a cut-free proof of A.

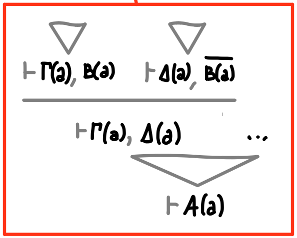
$$\begin{array}{c} | \\ \varphi \, \| \\ A \end{array}$$ is the given derivation

$$\parr\otimes \; \frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\lhd \; \frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

$\parr$, $\otimes$ ass., comm.

$\lhd$ ass., non-comm., self-dual

e.g., obtained from this sequent proof



Cut Elimination via Subatomic Projections in Propositional Logic with Decision Trees

$$\parr\otimes\ \frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\lhd\ \frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

$\parr$, $\otimes$ ass., comm.

$\lhd$ ass., non-comm., self-dual

Theorem Given a proof of A, we can build a cut-free proof of A.

$$\begin{array}{c} | \\ \varphi \Vert \\ A \end{array}$$ is the given derivation

and a an atom appearing in a cut instance

'subatomic'

$0a1 \leftarrow a$

$1a0 \leftarrow \bar{a}$

e.g., obtained from this sequent proof

$$\dfrac{\dfrac{\nabla}{\vdash \Gamma(a), B(a)} \qquad \dfrac{\nabla}{\vdash \Delta(a), \overline{B(a)}}}{\vdash \Gamma(a), \Delta(a) \quad \ldots}$$

$$\vdash A(a)$$

Cut Elimination via Subatomic Projections in Propositional Logic with Decision Trees

Theorem Given a proof of A, we can build a cut-free proof of A.

$$\begin{array}{c} 1 \\ \varphi \Big\| \\ A \end{array}$$ is the given derivation

and a an atom appearing in a cut instance

$$\vee\check{\wedge}\ \frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad \alpha a\ \frac{(A \,\alpha\, B)\ a\ (C \,\alpha\, D)}{(A\ a\ C)\ \alpha\ (B\ a\ D)}$$

$\vee$, $\wedge$ ass., comm.

a ass., non-comm., self-dual $\qquad \alpha \in \{\vee, \wedge, a, b, \dots\}$

'subatomic' $\quad 0a1 \leftarrow a$

$1a0 \leftarrow \bar{a}$

e.g., obtained from this sequent proof

$$\frac{\vdash \Gamma(a), B(a) \qquad \vdash \Delta(a), \overline{B(a)}}{\vdash \Gamma(a), \Delta(a) \qquad \dots}$$

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Cut Elimination via Subatomic Projections in Propositional Logic with Decision Trees

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$\varphi \Vert$ is the given derivation
A

and a an atom appearing in a cut instance

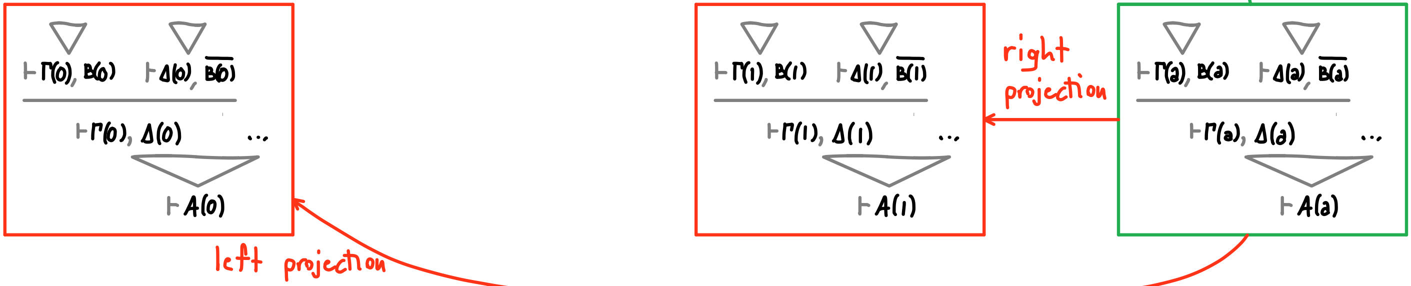
$$\vee\check{\wedge}\,\frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad \alpha a\,\frac{(A \,\alpha\, B)\, a\, (C \,\alpha\, D)}{(A \,a\, C)\, \alpha\, (B \,a\, D)}$$

$\vee$, $\wedge$ ass., comm.

a ass., non-comm., self-dual $\quad \alpha \in \{\vee, \wedge, a, b, \dots\}$

$0a1 \leftarrow a$

$1a0 \leftarrow \bar{a}$



Cut Elimination via Subatomic Projections in Propositional Logic with Decision Trees

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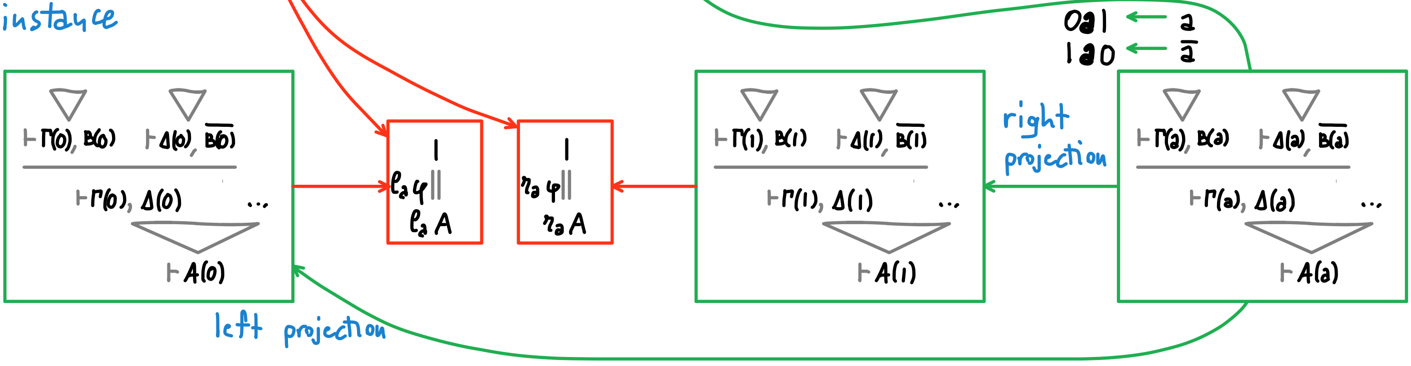
$\begin{array}{c} | \\ \varphi \| \\ A \end{array}$ is the given derivation

and a an atom appearing in a cut instance

$$\mathsf{v}\check{\wedge}\ \frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad \alpha\mathsf{a}\ \frac{(A \,\alpha\, B)\ \mathsf{a}\ (C \,\alpha\, D)}{(A\ \mathsf{a}\ C)\ \alpha\ (B\ \mathsf{a}\ D)}$$

$\vee$, $\wedge$ ass., comm.

a ass., non-comm., self-dual $\qquad \alpha \in \{\vee, \wedge, \mathsf{a}, \mathsf{b}, \ldots\}$



Cut Elimination via Subatomic Projections in Propositional Logic with Decision Trees

Theorem Given a proof of A, we can build a cut-free proof of A.

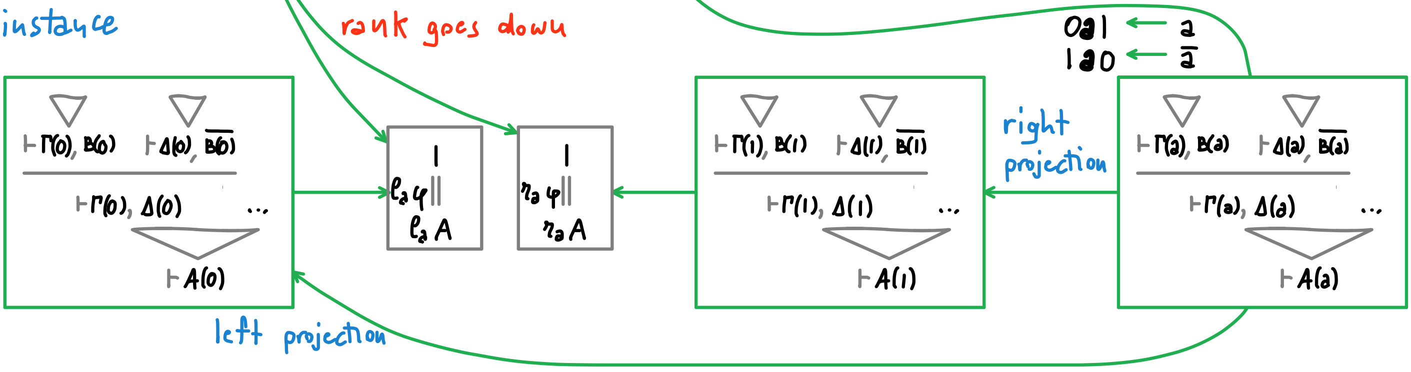
$$\varphi \, \overset{|}{\underset{A}{\|}}$$ is the given derivation

and $\mathbf{a}$ an atom appearing in a cut instance

$$\vee\check{\wedge}\ \frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad \alpha\mathbf{a}\ \frac{(A \,\alpha\, B)\ \mathbf{a}\ (C \,\alpha\, D)}{(A\ \mathbf{a}\ C)\ \alpha\ (B\ \mathbf{a}\ D)}$$

$\vee$, $\wedge$ ass., comm.

$\mathbf{a}$ ass., non-comm., self-dual $\qquad \alpha \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \ldots\}$



Cut Elimination via Subatomic Projections in Propositional Logic with Decision Trees

Theorem Given a proof of A, we can build a cut-free proof of A.

$$\begin{array}{c} 1 \\ \phi \Vert \\ A \end{array}$$ is the given derivation

and $\mathbf{a}$ an atom appearing in a cut instance

rank goes down

$$= \dfrac{1}{\boxed{\begin{array}{c} 1 \\ \ell_a \phi \Vert \\ \ell_a A \end{array}} \; \mathbf{a} \; \boxed{\begin{array}{c} 1 \\ r_a \phi \Vert \\ r_a A \end{array}}}$$

$$\boxed{\begin{array}{l} \vee\check{\wedge} \dfrac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad \alpha\mathbf{a} \dfrac{(A \,\alpha\, B)\, \mathbf{a}\, (C \,\alpha\, D)}{(A \,\mathbf{a}\, C)\, \alpha\, (B \,\mathbf{a}\, D)} \end{array}}$$

$\vee$, $\wedge$ ass., comm.

$\mathbf{a}$ ass., non-comm., self-dual $\qquad \alpha \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \ldots\}$

decision trees:

$$(1\,\mathbf{a}\,0)\,\mathbf{b}\,1 = \begin{cases} 1 & \text{if } b \\ \bar{a} & \text{if } \bar{b} \end{cases}$$

Cut Elimination via Subatomic Projections in Propositional Logic with Decision Trees

Theorem Given a proof of A, we can build a cut-free proof of A.

$$\begin{array}{c} 1 \\ \varphi \Vert \\ A \end{array}$$ is the given derivation

and $\mathbf{a}$ an atom appearing in a cut instance

rank goes down

$$= \frac{1}{\boxed{\begin{array}{c} 1 \\ \ell_a \varphi \Vert \\ \ell_a A \end{array}} \ \mathbf{a} \ \boxed{\begin{array}{c} 1 \\ r_a \varphi \Vert \\ r_a A \end{array}}} \qquad \begin{array}{c} \Vert \\ A \end{array}$$

no cuts

$$\vee\check{\wedge}\ \frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad \alpha\mathbf{a}\ \frac{(A \,\alpha\, B)\ \mathbf{a}\ (C \,\alpha\, D)}{(A\ \mathbf{a}\ C)\ \alpha\ (B\ \mathbf{a}\ D)}$$

$\vee$, $\wedge$ ass., comm.

$\mathbf{a}$ ass., non-comm., self-dual $\qquad \alpha \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \ldots\}$

repeated applications + contractions

decision trees:

$$(1\,\mathbf{a}\,0)\,\mathbf{b}\,1 = \begin{cases} 1 & \text{if } b \\ \bar{a} & \text{if } \bar{b} \end{cases}$$

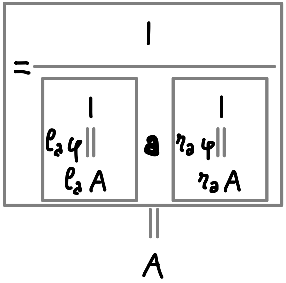
Cut Elimination via Subatomic Projections in Propositional Logic with Decision Trees

Theorem Given a proof of A, we can build a cut-free proof of A.

$$\begin{array}{c} 1 \\ \varphi \, \Vert \\ A \end{array}$$ is the given derivation

and $\mathbf{a}$ an atom appearing in a cut instance

rank goes down



This is free of cuts in $\mathbf{a}$. Repeat.

$$\vee\check{\wedge}\;\frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad \alpha\mathbf{a}\;\frac{(A \,\alpha\, B)\,\mathbf{a}\,(C \,\alpha\, D)}{(A \,\mathbf{a}\, C)\,\alpha\,(B \,\mathbf{a}\, D)}$$

$\vee$, $\wedge$ ass., comm.

$\mathbf{a}$ ass., non-comm., self-dual $\qquad \alpha \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \dots\}$

decision trees:

$$(1\,\mathbf{a}\,0)\,\mathbf{b}\,1 = \begin{cases} 1 & \text{if } b \\ \bar{a} & \text{if } \bar{b} \end{cases}$$

Cut Elimination via Subatomic Projections in Propositional Logic with Decision Trees

Theorem Given a proof of A, we can build a cut-free proof of A.

Proof If $\begin{array}{c}1\\ \varphi \Vert \\ A\end{array}$ is the given derivation and $\mathbf{a}$ an atom appearing in a cut instance, build

$$
\begin{array}{c}
= \dfrac{1}{\boxed{\begin{array}{c}1\\ \ell_a\varphi \Vert\\ \ell_a A\end{array}} \;\mathbf{a}\; \boxed{\begin{array}{c}1\\ r_a\varphi \Vert\\ r_a A\end{array}}} \\
\Vert \\
A
\end{array}
$$

This is free of cuts in $\mathbf{a}$. Repeat.

$$
\vee\check{\wedge}\,\frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad \alpha\mathbf{a}\,\frac{(A \,\alpha\, B)\,\mathbf{a}\,(C \,\alpha\, D)}{(A \,\mathbf{a}\, C)\,\alpha\,(B \,\mathbf{a}\, D)}
$$

$\vee$, $\wedge$ ass., comm.

$\mathbf{a}$ ass., non-comm., self-dual $\qquad \alpha \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \dots\}$

decision trees:

$$
(1\,\mathbf{a}\,0)\,\mathbf{b}\,1 = \begin{cases} 1 & \text{if } b \\ \bar{a} & \text{if } \bar{b} \end{cases}
$$

Cut Elimination via Subatomic Projections in Propositional Logic with Decision Trees

Theorem Given a proof of A, we can build a cut-free proof of A.

Proof If $\begin{array}{c}1\\ \varphi \,\Vert\\ A\end{array}$ is the given derivation and $\mathbf{a}$ an atom appearing in a cut instance, build

$$\begin{array}{c} = \dfrac{1}{\boxed{\begin{array}{c}1\\ \ell_a\varphi\,\Vert\\ \ell_a A\end{array}} \;\; \mathbf{a} \;\; \boxed{\begin{array}{c}1\\ r_a\varphi\,\Vert\\ r_a A\end{array}}} \\ \Vert \\ A \end{array}$$

This is free of cuts in $\mathbf{a}$. Repeat.

$$\vee\check{\wedge}\,\frac{(A\vee B)\wedge(C\vee D)}{(A\wedge C)\vee(B\vee D)} \qquad \alpha\mathbf{a}\,\frac{(A\,\alpha\,B)\,\mathbf{a}\,(C\,\alpha\,D)}{(A\,\mathbf{a}\,C)\,\alpha\,(B\,\mathbf{a}\,D)}$$

$\vee$, $\wedge$ ass., comm.

$\mathbf{a}$ ass., non-comm., self-dual $\qquad \alpha \in \{\vee,\wedge,\mathbf{a},\mathbf{b},\dots\}$

decision trees:

$$(1\,\mathbf{a}\,0)\,\mathbf{b}\,1 = \begin{cases} 1 & \text{if } b \\ \bar{a} & \text{if } \bar{b} \end{cases}$$

Adding decision trees is natural. What can we get?

Proof System = Subatomic Shape + Saturation + Unit Equations

[Aler Tubella, Guglielmi, ACM ToCL 2018]

[C. Barrett, Guglielmi, arXiv 2021]

+ papers in preparation

shape

$$\alpha\check{\beta}\ \frac{(A\ \alpha\ B)\ \beta\ (C\ \alpha\ D)}{(A\ \beta\ C)\ \alpha\ (B\ \check{\beta}\ D)} \qquad \alpha\hat{\beta}\ \frac{(A\ \beta\ B)\ \alpha\ (C\ \hat{\beta}\ D)}{(A\ \alpha\ C)\ \beta\ (B\ \alpha\ D)}$$

saturation

$\check{\vee} = \check{\wedge} = \vee$

$\hat{\vee} = \hat{\wedge} = \wedge$

$\check{\mathbf{a}} = \hat{\mathbf{a}} = \mathbf{a}$

unit equations

$$=\frac{1 \vee 1}{1} \qquad =\frac{0}{0 \wedge 0} \qquad \alpha \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \dots\}$$

$$=\frac{1}{1\ \mathbf{a}\ 1} \qquad =\frac{0\ \mathbf{a}\ 0}{0} \qquad =\frac{0}{0\ \mathbf{a}\ 0} \qquad =\frac{1\ \mathbf{a}\ 1}{1}$$

$$=\frac{A}{A \vee 0} \qquad =\frac{A \wedge 1}{A} \qquad =\frac{A \vee 0}{A} \qquad =\frac{A}{A \wedge 1}$$

+ mirror images

Adding decision trees is natural. What can we get?

A simple and natural proof system.

PROOF SYSTEM = SUBATOMIC SHAPE + SATURATION + UNIT EQUATIONS

shape

$$\alpha\check{\beta}\ \frac{(A\ \alpha\ B)\ \beta\ (C\ \alpha\ D)}{(A\ \beta\ C)\ \alpha\ (B\ \check{\beta}\ D)} \qquad \alpha\hat{\beta}\ \frac{(A\ \beta\ B)\ \alpha\ (C\ \hat{\beta}\ D)}{(A\ \alpha\ C)\ \beta\ (B\ \alpha\ D)}$$

$$\alpha \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \dots\}$$

saturation

$$\check{\vee} = \check{\wedge} = \vee$$

$$\hat{\vee} = \hat{\wedge} = \wedge$$

$$\check{\mathbf{a}} = \hat{\mathbf{a}} = \mathbf{a}$$

unit equations

$$=\frac{1 \vee 1}{1} \qquad =\frac{0}{0 \wedge 0}$$

$$=\frac{1}{1\,\mathbf{a}\,1} \qquad =\frac{0\,\mathbf{a}\,0}{0} \qquad =\frac{0}{0\,\mathbf{a}\,0} \qquad =\frac{1\,\mathbf{a}\,1}{1}$$

$$=\frac{A}{A \vee 0} \qquad =\frac{A \wedge 1}{A} \qquad =\frac{A \vee 0}{A} \qquad =\frac{A}{A \wedge 1}$$

+ mirror images

Proof System = Subatomic Shape + Saturation + Unit Equations

$$\wedge\hat{\wedge}\ \frac{(A\wedge B)\wedge(C\wedge D)}{(A\wedge C)\wedge(B\wedge D)}$$

commutativity / associativity

$$\vee\check{\wedge}\ \frac{(A\vee B)\wedge(C\vee D)}{(A\wedge C)\vee(B\vee D)}$$

switch (classical logic)

$$⅋\check{\otimes}\ \frac{(A ⅋ B)\otimes(C ⅋ D)}{(A\otimes C)⅋(B ⅋ D)}$$

switch (linear logic)

shape

$$\alpha\check{\beta}\ \frac{(A\,\alpha\,B)\,\beta\,(C\,\alpha\,D)}{(A\,\beta\,C)\,\alpha\,(B\,\check{\beta}\,D)} \qquad \alpha\hat{\beta}\ \frac{(A\,\beta\,B)\,\alpha\,(C\,\hat{\beta}\,D)}{(A\,\alpha\,C)\,\beta\,(B\,\alpha\,D)}$$

saturation

$$\check{\vee}=\check{\wedge}=\vee$$

$$\hat{\vee}=\hat{\wedge}=\wedge$$

$$\check{\mathbf{a}}=\hat{\mathbf{a}}=\mathbf{a}$$

$$\alpha\in\{\vee,\wedge,\mathbf{a},\mathbf{b},\dots\}$$

unit equations

$$=\frac{1\vee 1}{1} \qquad =\frac{0}{0\wedge 0}$$

$$=\frac{1}{1\,\mathbf{a}\,1} \qquad =\frac{0\,\mathbf{a}\,0}{0} \qquad =\frac{0}{0\,\mathbf{a}\,0} \qquad =\frac{1\,\mathbf{a}\,1}{1}$$

$$=\frac{A}{A\vee 0} \qquad =\frac{A\wedge 1}{A} \qquad =\frac{A\vee 0}{A} \qquad =\frac{A}{A\wedge 1}$$

+ mirror images

Proof System = Subatomic Shape + Saturation + Unit Equations

$$\wedge\hat{\wedge}\ \frac{(A \wedge B) \wedge (C \wedge D)}{(A \wedge C) \wedge (B \wedge D)}$$

commutativity / associativity

$$\vee\check{\wedge}\ \frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)}$$

switch (classical logic)

$$⅋\check{\otimes}\ \frac{(A ⅋ B) \otimes (C ⅋ D)}{(A \otimes C) ⅋ (B ⅋ D)}$$

switch (linear logic)

$$\wedge\hat{a}\ \frac{(0\ a\ 1) \wedge (1\ a\ 0)}{(0 \wedge 1)\ a\ (1 \wedge 0)} \rightarrow \frac{a \wedge \bar{a}}{0}$$

cut

shape

$$\alpha\check{\beta}\ \frac{(A \alpha B)\ \beta\ (C \alpha D)}{(A \beta C)\ \alpha\ (B \check{\beta} D)} \qquad \alpha\hat{\beta}\ \frac{(A \beta B)\ \alpha\ (C \hat{\beta} D)}{(A \alpha C)\ \beta\ (B \alpha D)}$$

saturation

$$\check{\vee} = \check{\wedge} = \vee$$

$$\hat{\vee} = \hat{\wedge} = \wedge$$

$$\check{a} = \hat{a} = a$$

unit equations

$$=\frac{1 \vee 1}{1} \qquad =\frac{0}{0 \wedge 0}$$

$\alpha \in \{\vee, \wedge, a, b, \ldots\}$

$$=\frac{1}{1\ a\ 1} \qquad =\frac{0\ a\ 0}{0} \qquad =\frac{0}{0\ a\ 0} \qquad =\frac{1\ a\ 1}{1}$$

$$=\frac{A}{A \vee 0} \qquad =\frac{A \wedge 1}{A} \qquad =\frac{A \vee 0}{A} \qquad =\frac{A}{A \wedge 1}$$

+ mirror images

Proof System = Subatomic Shape + Saturation + ~~Unit Equations~~

shape

$$\alpha\check{\beta}\,\frac{(A\,\alpha\,B)\,\beta\,(C\,\alpha\,D)}{(A\,\beta\,C)\,\alpha\,(B\,\check{\beta}\,D)} \qquad \alpha\hat{\beta}\,\frac{(A\,\beta\,B)\,\alpha\,(C\,\hat{\beta}\,D)}{(A\,\alpha\,C)\,\beta\,(B\,\alpha\,D)}$$

saturation

$$\check{\vee} = \check{\wedge} = \vee$$

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$$\check{\mathbf{a}} = \hat{\mathbf{a}} = \mathbf{a}$$

$$\alpha \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \ldots\}$$

unit equations

$$= \frac{1 \vee 1}{1} \qquad = \frac{0}{0 \wedge 0}$$

$$= \frac{1}{1\,\mathbf{a}\,1} \qquad = \frac{0\,\mathbf{a}\,0}{0} \qquad = \frac{0}{0\,\mathbf{a}\,0} \qquad = \frac{1\,\mathbf{a}\,1}{1}$$

$$= \frac{A}{A \vee 0} \qquad = \frac{A \wedge 1}{A} \qquad = \frac{A \vee 0}{A} \qquad = \frac{A}{A \wedge 1}$$

+ mirror images

Work in progress with V. Barrett.

Proof System = Subatomic Shape + Saturation

Lemma (Eversion)

Given any pure formulae $\mathbf{A}$ and $\mathbf{B}$, there exist

$$[\mathbf{B}^i \Rightarrow x_i]_{\underline{\check{\mathbf{A}}}} \check{\mathbf{A}} \overset{\star\check{\star}}{=} [\check{\mathbf{A}}^j \Rightarrow y_j]_{\underline{\mathbf{B}}} \mathbf{B}$$

where the $\mathbf{B}^i$s (resp. $\check{\mathbf{A}}^j$s) are renamings of $\mathbf{B}$ (resp. $\check{\mathbf{A}}$) and $\bigcup_i \underline{\mathbf{B}^i} = \bigcup_j \underline{\check{\mathbf{A}}^j}$.

Very powerful!

$$\frac{(A \beta B) \alpha (C \beta' D)}{(A \alpha C) \beta (B \alpha' D)}$$

$\check{\vee} = \check{\wedge} = \vee$

$\hat{\vee} = \hat{\wedge} = \wedge$

$\check{\mathbf{a}} = \hat{\mathbf{a}} = \mathbf{a}$

$\alpha \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \ldots\}$

Work in progress with V. Barrett

PROOF SYSTEM = SUBATOMIC SHAPE + SATURATION

Lemma (Eversion)

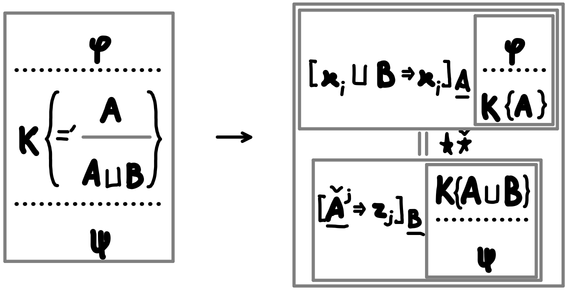
Given any pure formulae $\mathbf{A}$ and $\mathbf{B}$, there exist

$$[\mathbf{B}^i \Rightarrow x_i]_{\underline{\check{\mathbf{A}}}}\,\check{\mathbf{A}} \;\overset{\star\check{\star}}{\|}\; [\check{\mathbf{A}}^j \Rightarrow y_j]_{\underline{\mathbf{B}}}\,\mathbf{B}$$

where the $\mathbf{B}^i$'s (resp. $\check{\mathbf{A}}^j$'s) are renamings of $\mathbf{B}$ (resp. $\check{\mathbf{A}}$) and $\bigcup_i \underline{\mathbf{B}^i} = \bigcup_j \underline{\check{\mathbf{A}}^j}$.

Idea

Completeness for classical logic



$$\frac{(A\,\beta\,B)\;\alpha\;(C\,\beta'\,D)}{(A\,\alpha\,C)\;\beta\;(B\,\alpha'\,D)}$$

$\check{\vee} = \check{\wedge} = \vee$

$\hat{\vee} = \hat{\wedge} = \wedge$

$\check{a} = \hat{a} = a$

$\alpha \in \{\vee, \wedge, a, b, \ldots\}$

Work in progress with V. Barrett

Proof System = Subatomic Shape + Saturation

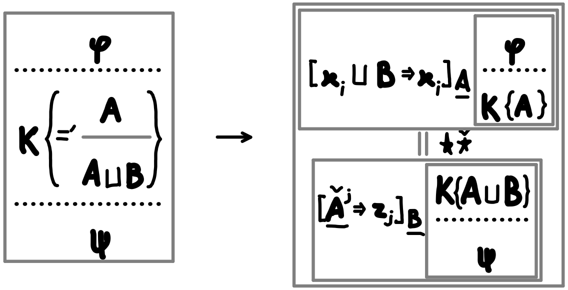
Lemma (Eversion)

Given any pure formulae **A** and **B**, there exist

$$\begin{array}{c}[\mathbf{B}^i \Rightarrow x_i]_{\check{\underline{\mathbf{A}}}}\check{\mathbf{A}} \\ \| \star\check{\star} \\ [\check{\mathbf{A}}^j \Rightarrow y_j]_{\underline{\mathbf{B}}}\mathbf{B}\end{array}$$

where the $\mathbf{B}^i$'s (resp. $\check{\mathbf{A}}^j$'s) are renamings of **B** (resp. $\check{\mathbf{A}}$) and $\bigcup_i \underline{\mathbf{B}^i} = \bigcup_j \underline{\check{\mathbf{A}}^j}$.

Idea Completeness for classical logic



$$\frac{(A \,\beta\, B)\ \alpha\ (C \,\beta'\, D)}{(A \,\alpha\, C)\ \beta\ (B \,\alpha'\, D)}$$

$\check{\vee} = \check{\wedge} = \vee$

$\hat{\vee} = \hat{\wedge} = \wedge$

$\check{\mathbf{a}} = \hat{\mathbf{a}} = \mathbf{a}$

$\alpha \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \ldots\}$

- all the structure needed for normalisation is here
- linear (totally!)
- normalisation at the standard level can be recovered from the subatomic one

Work in progress with V. Barrett

PROOF SYSTEM = SUBATOMIC SHAPE + SATURATION

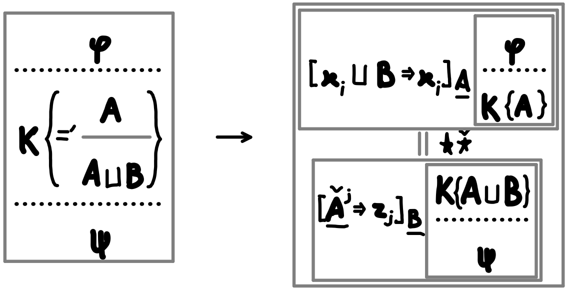
Lemma (Eversion)

Given any pure formulae **A** and **B**, there exist

$$[\mathbf{B}^i \Rightarrow x_i]_{\underline{\check{\mathbf{A}}}}\check{\mathbf{A}} \overset{\star\check{\star}}{=} [\check{\mathbf{A}}^j \Rightarrow y_j]_{\underline{\mathbf{B}}}\mathbf{B}\ .$$

where the $\mathbf{B}^i$'s (resp. $\check{\mathbf{A}}^j$'s) are renamings of **B** (resp. $\check{\mathbf{A}}$) and $\bigsqcup_i \underline{\mathbf{B}^i} = \bigsqcup_j \underline{\check{\mathbf{A}}^j}$.

Idea Completeness for classical logic



$$\frac{(A\,\beta\,B)\,\alpha\,(C\,\beta'\,D)}{(A\,\alpha\,C)\,\beta\,(B\,\alpha'\,D)}$$

$$\alpha \in \{\vee, \wedge, \mathsf{a}, \mathsf{b}, \ldots\}$$

$$\check{\vee} = \check{\wedge} = \vee$$

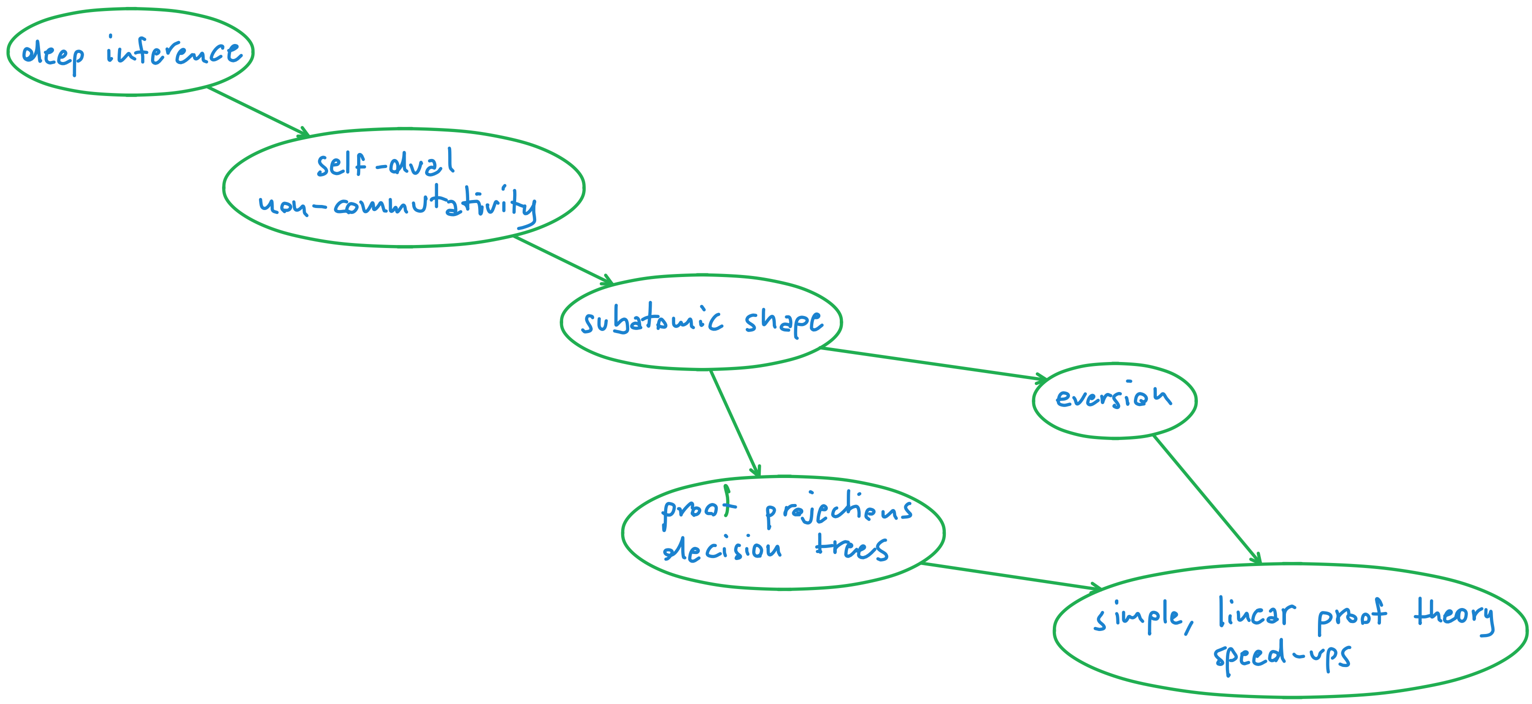
$$\hat{\vee} = \hat{\wedge} = \wedge$$

$$\check{\mathsf{a}} = \hat{\mathsf{a}} = \mathsf{a}$$

- all the structure needed for normalisation is here
- linear (totally!)
- normalisation at the standard level can be recovered from the subatomic one

This is being extended to other logics and higher orders.

Summary


deep inference
self-dual non-commutativity
subatomic shape
eversion
proof projections decision trees
simple, linear proof theory speed-ups

STATMAN TAUTOLOGIES

Definition We call Statman tautologies the formulae $S_1, S_2, \ldots,$ where a_i and b_i stand for $(0\, \mathbf{a}_i\, 1)$ and $(0\, \mathbf{b}_i\, 1)$:

$$S_1 \equiv (\bar{a}_1 \wedge \bar{b}_1) \vee (a_1 \vee b_1) \qquad \text{for } n > k \geq 1$$

$$S_n \equiv$$
$$(\bar{a}_n \wedge \bar{b}_n) \vee ((A^n_{n-1} \wedge B^n_{n-1}) \vee \cdots \vee (A^n_1 \wedge B^n_1)) \vee (a_1 \vee b_1)$$

where:

$$A^n_k \equiv (a_n \vee b_n) \wedge \cdots \wedge (a_{k+1} \vee b_{k+1}) \wedge \bar{a}_k$$
$$(\equiv (a_n \vee b_n) \wedge A^{n-1}_k \quad \text{if } n-1 > k)$$

$$B^n_k \equiv (a_n \vee b_n) \wedge \cdots \wedge (a_{k+1} \vee b_{k+1}) \wedge \bar{b}_k$$
$$(\equiv (a_n \vee b_n) \wedge B^{n-1}_k \quad \text{if } n-1 > k)$$

We work modulo associativity.

Examples

$$S_1 \equiv (\bar{a}_1 \wedge \bar{b}_1) \vee (a_1 \vee b_1)$$

$$\begin{aligned} S_2 \equiv\ & (\bar{a}_2 \wedge \bar{b}_2) \\ & \vee (((a_2 \vee b_2) \wedge \bar{a}_1) \wedge \\ & \quad ((a_2 \vee b_2) \wedge \bar{b}_1) \quad) \\ & \vee \qquad (a_1 \vee b_1) \end{aligned}$$

$$\begin{aligned} S_3 \equiv\ & (\bar{a}_3 \wedge \bar{b}_3) \\ & \vee (((a_3 \vee b_3) \wedge \bar{a}_2) \wedge \\ & \quad ((a_3 \vee b_3) \wedge \bar{b}_2) \quad) \\ & \vee (((a_3 \vee b_3) \wedge (a_2 \vee b_2) \wedge \bar{a}_1) \wedge \\ & \quad ((a_3 \vee b_3) \wedge (a_2 \vee b_2) \wedge \bar{b}_1) \quad) \\ & \vee \qquad (a_1 \vee b_1) \end{aligned}$$

In cut-free Gentzen systems, all proofs of Statman tautologies grow at least exponentially [*].

[*] Statman, R. (1978). Bounds for proof-search and speed-up in the predicate calculus. Ann. Math. Logic 15, 225–287.

Statman Tautologies

Theorem There exist cut-free proofs of Statman tautologies of size $O(m^{2.5})$ on the size m of the tautologies.

$S_1 \equiv (\bar{a}_1 \wedge \bar{b}_1) \vee (a_1 \vee b_1)$ for $n > k \geq 1$

$S_n \equiv$

$(\bar{a}_n \wedge \bar{b}_n) \vee ((A^n_{n-1} \wedge B^n_{n-1}) \vee \cdots \vee (A^n_1 \wedge B^n_1)) \vee (a_1 \vee b_1)$

where: $A^n_k \equiv (a_n \vee b_n) \wedge \cdots \wedge (a_{k+1} \vee b_{k+1}) \wedge \bar{a}_k$

$(\equiv (a_n \vee b_n) \wedge A^{n-1}_k$ if $n-1 > k)$

$B^n_k \equiv (a_n \vee b_n) \wedge \cdots \wedge (a_{k+1} \vee b_{k+1}) \wedge \bar{b}_k$

$(\equiv (a_n \vee b_n) \wedge B^{n-1}_k$ if $n-1 > k)$

Proof We build a cut-free derivation

$$\begin{array}{c} 1 \\ \| \\ S_1 \\ \| \\ \vdots \\ \| \\ S_n \end{array}$$

Statman Tautologies

Theorem There exist cut-free proofs of Statman tautologies of size $O(m^{2.5})$ on the size m of the tautologies.

$S_1 \equiv (\bar{a}_1 \wedge \bar{b}_1) \vee (a_1 \vee b_1)$ $O(m)$ for $n > k \geq 1$

$$\boxed{\begin{array}{l} S_n \equiv \qquad O(n^2) \\ (\bar{a}_n \wedge \bar{b}_n) \vee ((A^n_{n-1} \wedge B^n_{n-1}) \vee \dots \vee (A^n_1 \wedge B^n_1)) \vee (a_1 \vee b_1) \end{array}}$$

where: $A^n_k \equiv (a_n \vee b_n) \wedge \dots \wedge (a_{k+1} \vee b_{k+1}) \wedge \bar{a}_k$

$(\equiv (a_n \vee b_n) \wedge A^{n-1}_k$ if $n-1 > k)$

$B^n_k \equiv (a_n \vee b_n) \wedge \dots \wedge (a_{k+1} \vee b_{k+1}) \wedge \bar{b}_k$

$(\equiv (a_n \vee b_n) \wedge B^{n-1}_k$ if $n-1 > k)$

Proof We build a cut-free derivation

$$O(m^{0.5}) \text{ steps} \left\{ \begin{array}{c} \vdots \\ \| \\ S_1 \\ \| \\ \vdots \\ \| \\ S_n \end{array} \right.$$

Statman Tautologies

Theorem There exist cut-free proofs of Statman tautologies of size $O(m^{2.5})$ on the size m of the tautologies.

$S_1 \equiv (\bar{a}_1 \wedge \bar{b}_1) \vee (a_1 \vee b_1)$ for $n > k \geq 1$

$S_n \equiv$

$(\bar{a}_n \wedge \bar{b}_n) \vee ((A^n_{n-1} \wedge B^n_{n-1}) \vee \cdots \vee (A^n_1 \wedge B^n_1)) \vee (a_1 \vee b_1)$

where: $A^n_k \equiv (a_n \vee b_n) \wedge \cdots \wedge (a_{k+1} \vee b_{k+1}) \wedge \bar{a}_k$

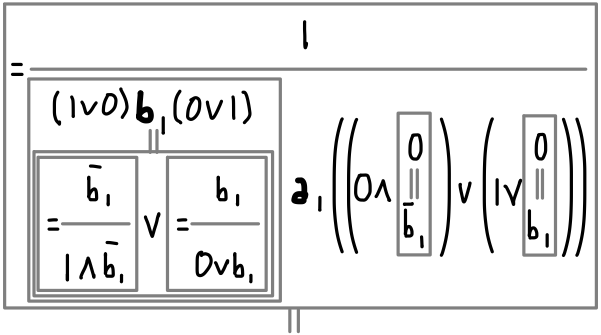
$(\equiv (a_n \vee b_n) \wedge A^{n-1}_k$ if $n-1 > k)$

$B^n_k \equiv (a_n \vee b_n) \wedge \cdots \wedge (a_{k+1} \vee b_{k+1}) \wedge \bar{b}_k$

$(\equiv (a_n \vee b_n) \wedge B^{n-1}_k$ if $n-1 > k)$

Proof

Base case



$$(\bar{a}_1 \wedge \bar{b}_1) \vee (a_1 \vee b_1)$$

Statman Tautologies

Theorem There exist cut-free proofs of Statman tautologies of size $O(m^{2.5})$ on the size m of the tautologies.

$S_1 \equiv (\bar a_1 \wedge \bar b_1) \vee (a_1 \vee b_1)$ for $n > k \geq 1$

$S_n \equiv$

$(\bar a_n \wedge \bar b_n) \vee ((A^n_{n-1} \wedge B^n_{n-1}) \vee \cdots \vee (A^n_1 \wedge B^n_1)) \vee (a_1 \vee b_1)$

where: $A^n_k \equiv (a_n \vee b_n) \wedge \cdots \wedge (a_{k+1} \vee b_{k+1}) \wedge \bar a_k$

$(\equiv (a_n \vee b_n) \wedge A^{n-1}_k$ if $n-1 > k)$

$B^n_k \equiv (a_n \vee b_n) \wedge \cdots \wedge (a_{k+1} \vee b_{k+1}) \wedge \bar b_k$

$(\equiv (a_n \vee b_n) \wedge B^{n-1}_k$ if $n-1 > k)$

Proof

Inductive step

$$\begin{array}{c} S_{n-1} \\ \| \\ \boxed{\begin{array}{c} \boxed{\begin{array}{c} S_{n-1} \\ \| \\ \boxed{\boxed{=\dfrac{1}{\varphi}}\; b_n \; \boxed{=\dfrac{S_{n-1}}{0 \vee (1 \wedge \bar a_{n-1} \wedge 1 \wedge \bar b_{n-1}) \vee ((1 \wedge A^{n-1}_{n-2} \wedge 1 \wedge B^{n-1}_{n-2}) \vee \cdots \vee (1 \wedge A^{n-1}_1 \wedge 1 \wedge B^{n-1}_1) \vee (a_1 \vee b_1)}}} \\ \| \\ \boxed{\begin{array}{c} \left(\boxed{=\dfrac{\bar b_n}{1 \wedge \bar b_n}} \vee \left(\left(\boxed{=\dfrac{b_n}{0 \vee b_n}} \wedge \bar a_{n-1} \wedge \boxed{=\dfrac{b_n}{0 \vee b_n}} \wedge \bar b_{n-1}\right) \vee \cdots \vee \left(\boxed{=\dfrac{b_n}{0 \vee b_n}} \wedge A^{n-1}_1 \wedge \boxed{=\dfrac{b_n}{0 \vee b_n}} \wedge B^{n-1}_1\right)\right) \vee (a_1 \vee b_1)\right) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \ell_{a_n} S_n \end{array}} \end{array}} \; a_n \; \boxed{\begin{array}{c} S_{n-1} \\ \psi \, \| \\ r_{a_n} S_n \end{array}} \end{array}} \\ \| \\ S_n \end{array}$$

where:

$$\varphi \equiv 1 \vee \boxed{\begin{array}{c} 0 \\ \| \\ (0 \wedge \bar a_{n-1} \wedge 0 \wedge \bar b_{n-1}) \vee ((0 \wedge A^{n-1}_{n-2} \wedge 0 \wedge B^{n-1}_{n-2}) \vee \cdots \vee (0 \wedge A^{n-1}_1 \wedge 0 \wedge B^{n-1}_1)) \vee (a_1 \vee b_1) \end{array}}$$

$$\psi \equiv \boxed{= \dfrac{(\bar a_{n-1} \wedge \bar b_{n-1}) \vee ((A^{n-1}_{n-2} \wedge B^{n-1}_{n-2}) \vee \cdots \vee (A^{n-1}_1 \wedge B^{n-1}_1)) \vee (a_1 \vee b_1)}{\left(0 \wedge \boxed{\begin{array}{c}0\\ \| \\ \bar b_n\end{array}}\right) \vee \left(\left(\left(1 \vee \boxed{\begin{array}{c}0\\ \| \\ b_n\end{array}}\right) \wedge \bar a_{n-1} \wedge \left(1 \vee \boxed{\begin{array}{c}0\\ \| \\ b_n\end{array}}\right) \wedge \bar b_{n-1}\right) \vee \cdots \vee \left(\left(1 \vee \boxed{\begin{array}{c}0\\ \| \\ b_n\end{array}}\right) \wedge A^{n-1}_1 \wedge \left(1 \vee \boxed{\begin{array}{c}0\\ \| \\ b_n\end{array}}\right) \wedge B^{n-1}_1\right) \vee (a_1 \vee b_1)}}$$

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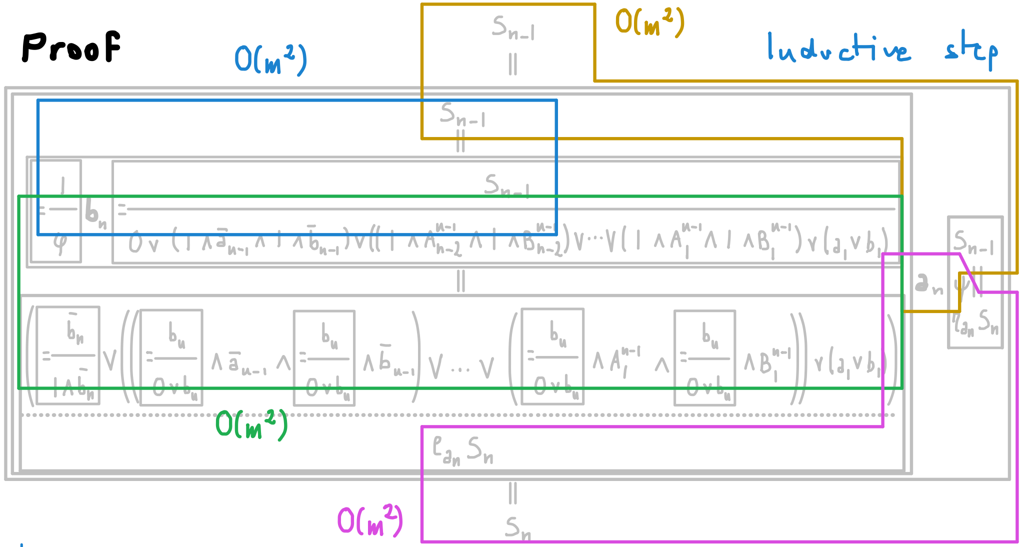
$(\equiv (a_n \vee b_n) \wedge B^{n-1}_k$ if $n-1 > k)$

Proof



where:

$$\varphi \equiv 1 \vee \boxed{\begin{array}{c} 0 \\ \| \\ (0 \wedge \bar{a}_{n-1} \wedge 0 \wedge \bar{b}_{n-1}) \vee ((0 \wedge A^{n-1}_{n-2} \wedge 0 \wedge B^{n-1}_{n-2}) \vee \cdots \vee (0 \wedge A^{n-1}_1 \wedge 0 \wedge B^{n-1}_1)) \vee (a_1 \vee b_1) \end{array}}$$

$$\psi \equiv \boxed{= \dfrac{(\bar{a}_{n-1} \wedge \bar{b}_{n-1}) \vee ((A^{n-1}_{n-2} \wedge B^{n-1}_{n-2}) \vee \cdots \vee (A^{n-1}_1 \wedge B^{n-1}_1)) \vee (a_1 \vee b_1)}{\left(0 \wedge \boxed{\begin{array}{c} 0 \\ \| \\ \bar{b}_n \end{array}}\right) \vee \left(\left(\left(1 \vee \boxed{\begin{array}{c} 0 \\ \| \\ b_n \end{array}}\right) \wedge \bar{a}_{n-1} \wedge \left(1 \vee \boxed{\begin{array}{c} 0 \\ \| \\ b_n \end{array}}\right) \wedge \bar{b}_{n-1}\right) \vee \cdots \vee \left(\left(1 \vee \boxed{\begin{array}{c} 0 \\ \| \\ b_n \end{array}}\right) \wedge A^{n-1}_1 \wedge \left(1 \vee \boxed{\begin{array}{c} 0 \\ \| \\ b_n \end{array}}\right) \wedge B^{n-1}_1\right)\right) \vee (a_1 \vee b_1)}}$$