Quantaloidal approach to constraint satisfaction

Soichiro Fujii, Yuni Iwamasa and Kei Kimura

ACT 2021

Quantaloids

= {complete join-semilattices}-enriched categories

Quantaloidal approach to constraint satisfaction

Constraint satisfaction problem (CSP):

general framework for computational problems

including k-SAT, graph k-colouring, …

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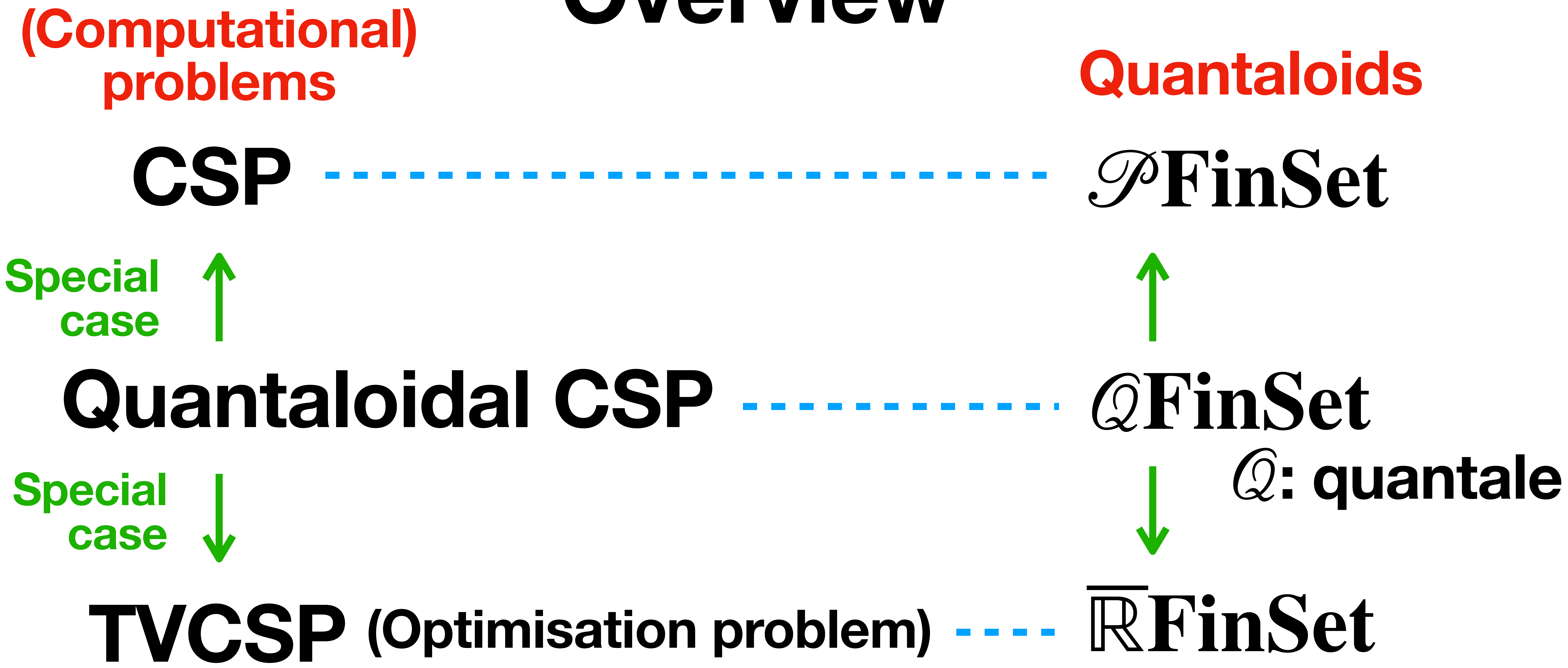
Overview


(Computational) problems
Quantaloids
CSP
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Special case
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𝒬: quantale
Special case
TVCSP (Optimisation problem)
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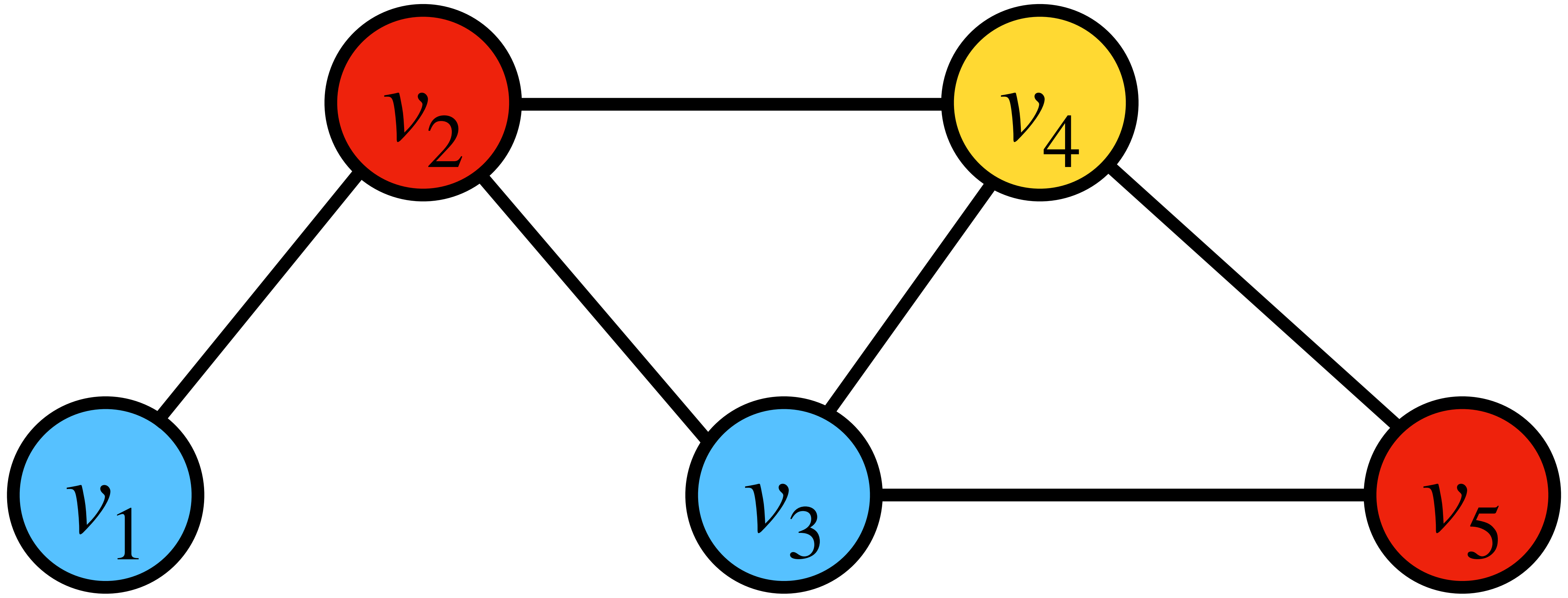
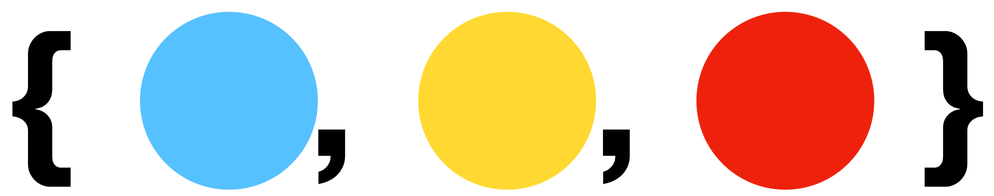
Graph k-colouring ($k \in \mathbb{N}$)



$\exists s : \{v_1, \ldots, v_5\} \to \{1, \ldots, k\}$ s.t. $\forall$edge $(v_i, v_j), s(v_i) \neq s(v_j)$?

Ex. k **= 3**



A CSP instance $I = (V, D, \mathscr{C})$ consists of:

- V: finite set of variables
- D: finite set called the domain
- $\mathscr{C}$: finite set of "constraints"

A constraint is $(k, \mathbf{x}, \rho)$ where

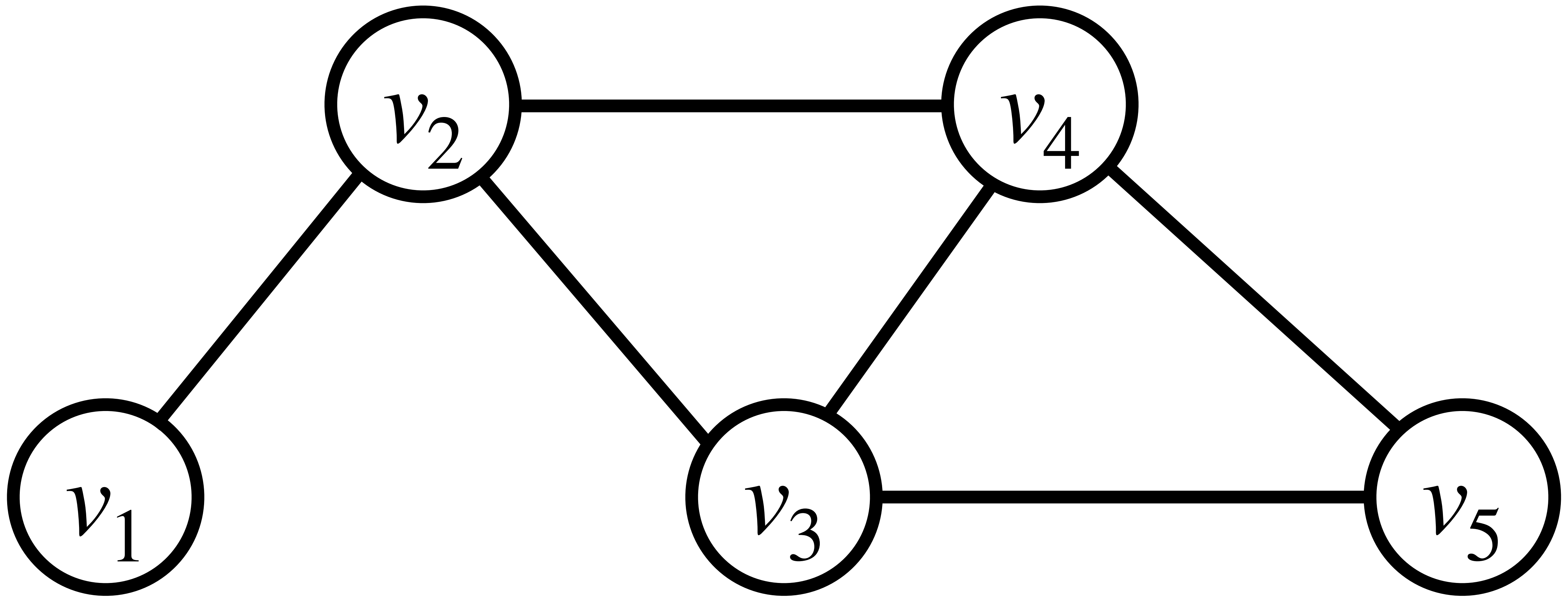
- $k \in \mathbb{N}$, $\mathbf{x} \in V^k$, $\rho \subseteq D^k$.

A function $s : V \to D$ satisfies the constraint $(k, \mathbf{x} = (x_1, \ldots, x_k), \rho)$ if $(s(x_1), \ldots, s(x_k)) \in \rho$.

A solution of $I = (V, D, \mathscr{C})$ is a function $s : V \to D$ satisfying every constraint in $\mathscr{C}$.

$\mathscr{S}(I) = \{$solutions of $I\} \subseteq [V, D]$

Ex. Graph k-colouring



$\exists s \colon \{v_1, \ldots, v_5\} \to \{1, \ldots, k\}$ s.t. $\forall$edge $(v_i, v_j), s(v_i) \neq s(v_j)$?

A function $s \colon V \to D$ **satisfies** the constraint $(k', \mathbf{x} = (x_1, \ldots, x_{k'}), \rho)$ if $(s(x_1), \ldots, s(x_{k'})) \in \rho$.

$V = \{v_1, \ldots, v_5\}$
$D = \{1, \ldots, k\}$
$\mathscr{C} = \{(2, (v_i, v_j), \neq \subseteq D^2) \mid (v_i, v_j) \colon$ **edge**$\}$

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Quantaloids

CSP ---- $\mathscr{P}\mathbf{FinSet}$

Special case ↑

Quantaloidal CSP ---- $\mathscr{Q}\mathbf{FinSet}$

$\mathscr{Q}$: quantale

Special case ↓

TVCSP (Optimisation problem) ---- $\overline{\mathbb{R}}\mathbf{FinSet}$

The 2-category $\mathscr{P}\mathbf{FinSet}$:

Obj. Finite sets

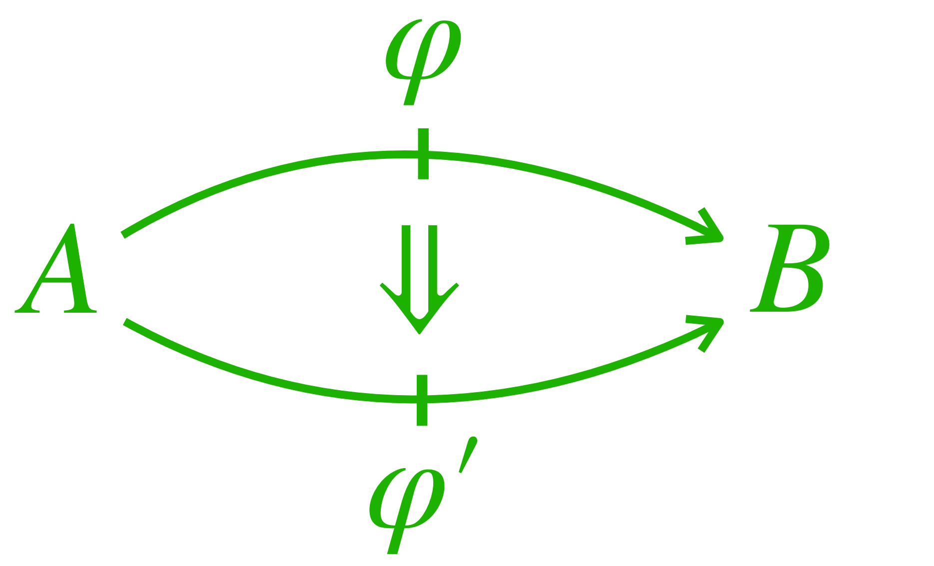
Mor.
$$\frac{A \xrightarrow{\varphi} B}{\varphi \subseteq [A, B]}$$

Comp. $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$

$$\psi \circ \varphi = \{\, g \circ f \mid g \in \psi, f \in \varphi \,\}$$

Id. $A \xrightarrow{\{\mathrm{id}_A\}} A$

2-cell



$$\varphi \subseteq \varphi'$$

$\mathscr{P}\mathbf{FinSet}$ is a quantaloid (the free quantaloid over $\mathbf{FinSet}$):

- $\forall A, B \in \mathscr{P}\mathbf{FinSet}$, $\mathscr{P}\mathbf{FinSet}(A, B) = (\mathscr{P}[A, B], \subseteq)$ is a complete lattice.
- $\forall A, B, C \in \mathscr{P}\mathbf{FinSet}$,
$$\mathscr{P}\mathbf{FinSet}(B, C) \times \mathscr{P}\mathbf{FinSet}(A, B) \xrightarrow{\circ} \mathscr{P}\mathbf{FinSet}(A, C)$$
preserves arbitrary joins in each variable:

$B \xrightarrow{\psi} C \quad (A \xrightarrow{\varphi_i} B)_{i \in I}$

$$\psi \circ \Big(\bigvee_{i \in I} \varphi_i\Big) = \bigvee_{i \in I} (\psi \circ \varphi_i)$$

$(B \xrightarrow{\psi_i} C)_{i \in I} \quad A \xrightarrow{\varphi} B$

$$\Big(\bigvee_{i \in I} \psi_i\Big) \circ \varphi = \bigvee_{i \in I} (\psi_i \circ \varphi)$$

In particular,

- $\forall A \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B, C \in \mathscr{P}\mathbf{FinSet},$

$$\mathscr{P}\mathbf{FinSet}(\varphi, C)\colon \mathscr{P}\mathbf{FinSet}(B, C) \longrightarrow \mathscr{P}\mathbf{FinSet}(A, C)$$

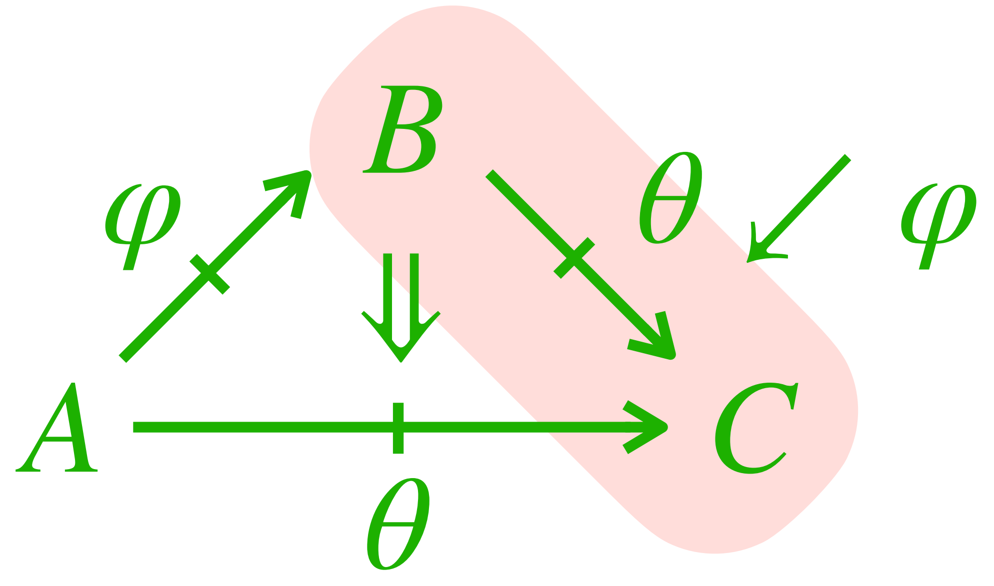
$$(B \overset{\psi}{\longrightarrow\!\!\!\!\!\!\!+\;\;} C) \longmapsto (A \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B \overset{\psi}{\longrightarrow\!\!\!\!\!\!\!+\;\;} C)$$

preserves arbitrary joins.

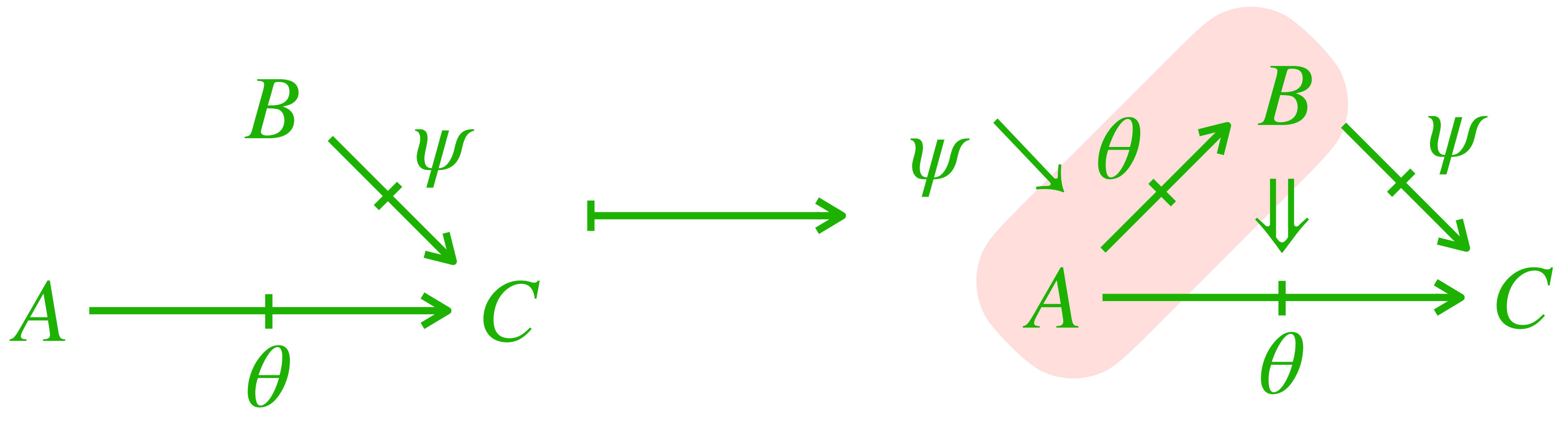
$\Longleftrightarrow$ $\mathscr{P}\mathbf{FinSet}(\varphi, C)$ has a right adjoint

$$(-) \swarrow \varphi \colon \mathscr{P}\mathbf{FinSet}(A, C) \longrightarrow \mathscr{P}\mathbf{FinSet}(B, C)$$

$$(A \overset{\theta}{\longrightarrow\!\!\!\!\!\!\!+\;\;} C) \longmapsto$$

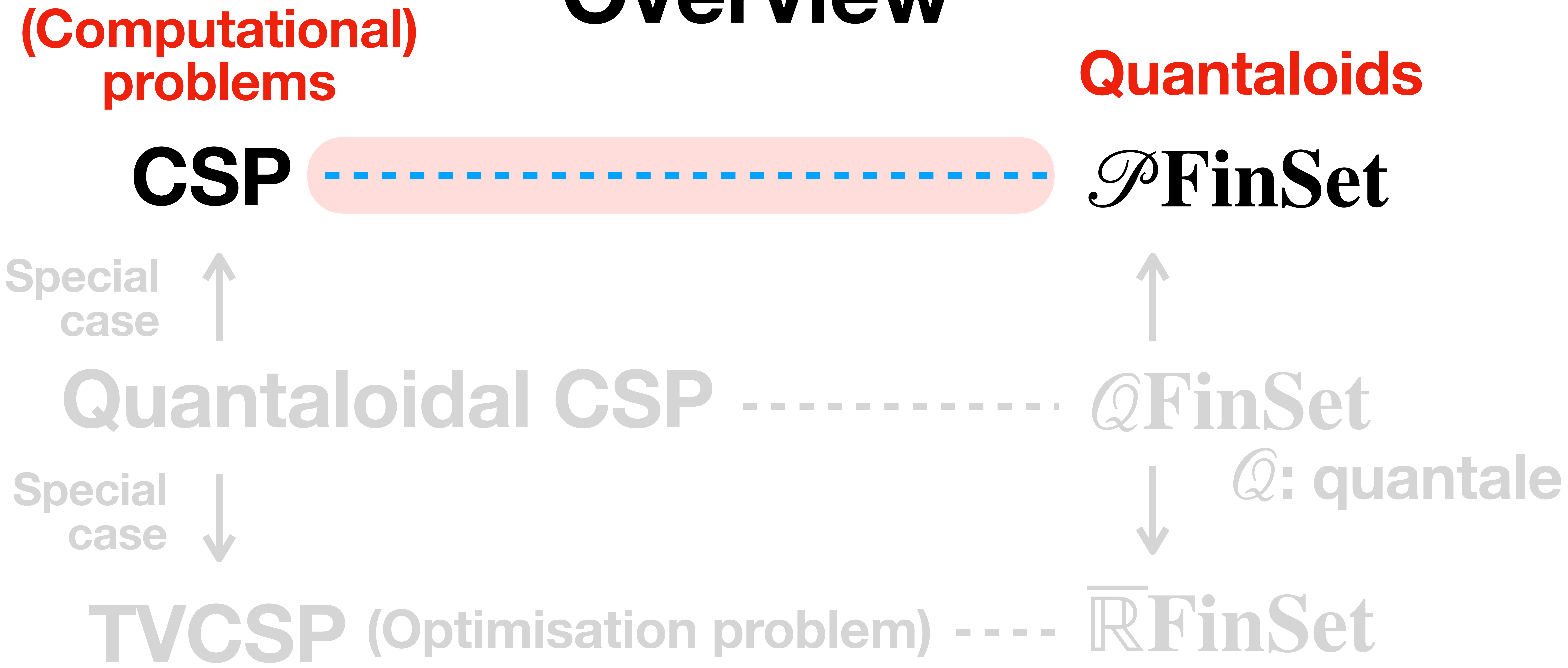


The right extension of θ along φ



The right lifting of θ along ψ

Overview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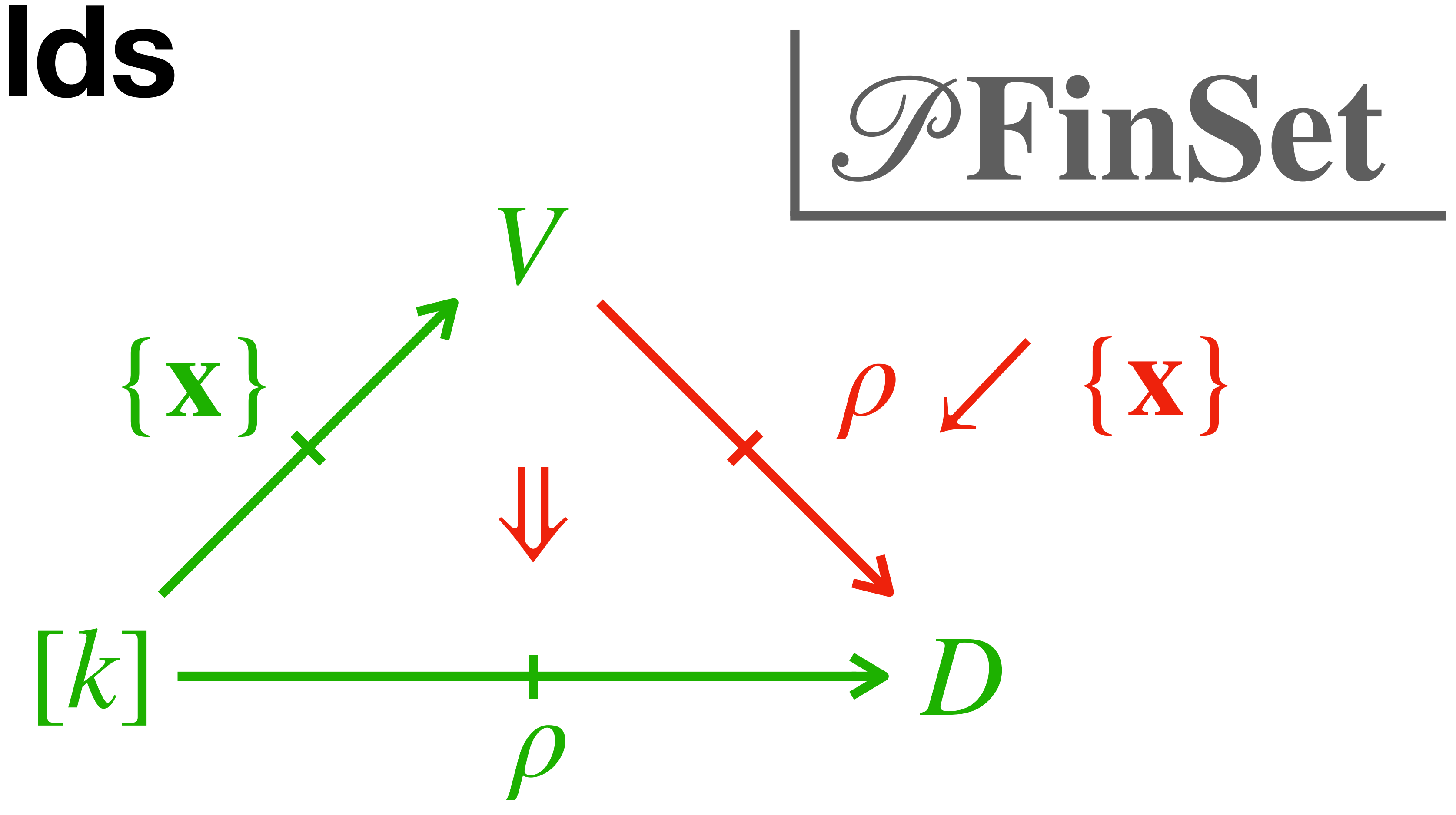
Each constraint $(k, \mathbf{x}, \rho)$ yields



$$\rho \swarrow \{\mathbf{x}\} \subseteq [V, D]$$

$$\| $$

$\{ s \colon V \to D \mid s$ satisfies the constraint $(k, \mathbf{x}, \rho) \}$

A **CSP instance** $I = (V, D, \mathscr{C})$ consists of:

- V: finite set of **variables**
- D: finite set called the **domain**
- $\mathscr{C}$: finite set of "constraints"

A **constraint** is $(k, \mathbf{x}, \rho)$ where

- $k \in \mathbb{N}$, $\quad \mathbf{x} \in V^k$, $\quad \rho \subseteq D^k$.

A function $s \colon V \to D$ **satisfies** the constraint $(k, \mathbf{x} = (x_1, \ldots, x_k), \rho)$ if $(s(x_1), \ldots, s(x_k)) \in \rho$.

A **solution** of $I = (V, D, \mathscr{C})$ is a function $s \colon V \to D$ satisfying every constraint in $\mathscr{C}$.

$$\mathcal{S}(I) = \{\text{solutions of } I\} \subseteq [V, D]$$

$$\mathcal{S}(I) = \bigcap_{(k,\mathbf{x},\rho)\in\mathscr{C}} \rho \swarrow \{\mathbf{x}\} \colon V \longrightarrow\!\!\!\!\!\!\!+\;\; D$$

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(Computational) problems

Quantaloids

CSP - - - - - - - - - - $\mathscr{P}\mathbf{FinSet}$

Special case ↑ ↑

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$\mathscr{Q}$: quantale

Special case ↓ ↓

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Polymorphisms

Special case ↑ ↑

Quantaloidal CSP - - - - - $\mathscr{Q}$FinSet

Special case ↓ $\mathscr{Q}$-valued polymorphisms ↓ $\mathscr{Q}$: quantale

TVCSP (Optimisation problem) - - - - $\overline{\mathbb{R}}$FinSet

$\overline{\mathbb{R}}$-valued polymorphisms

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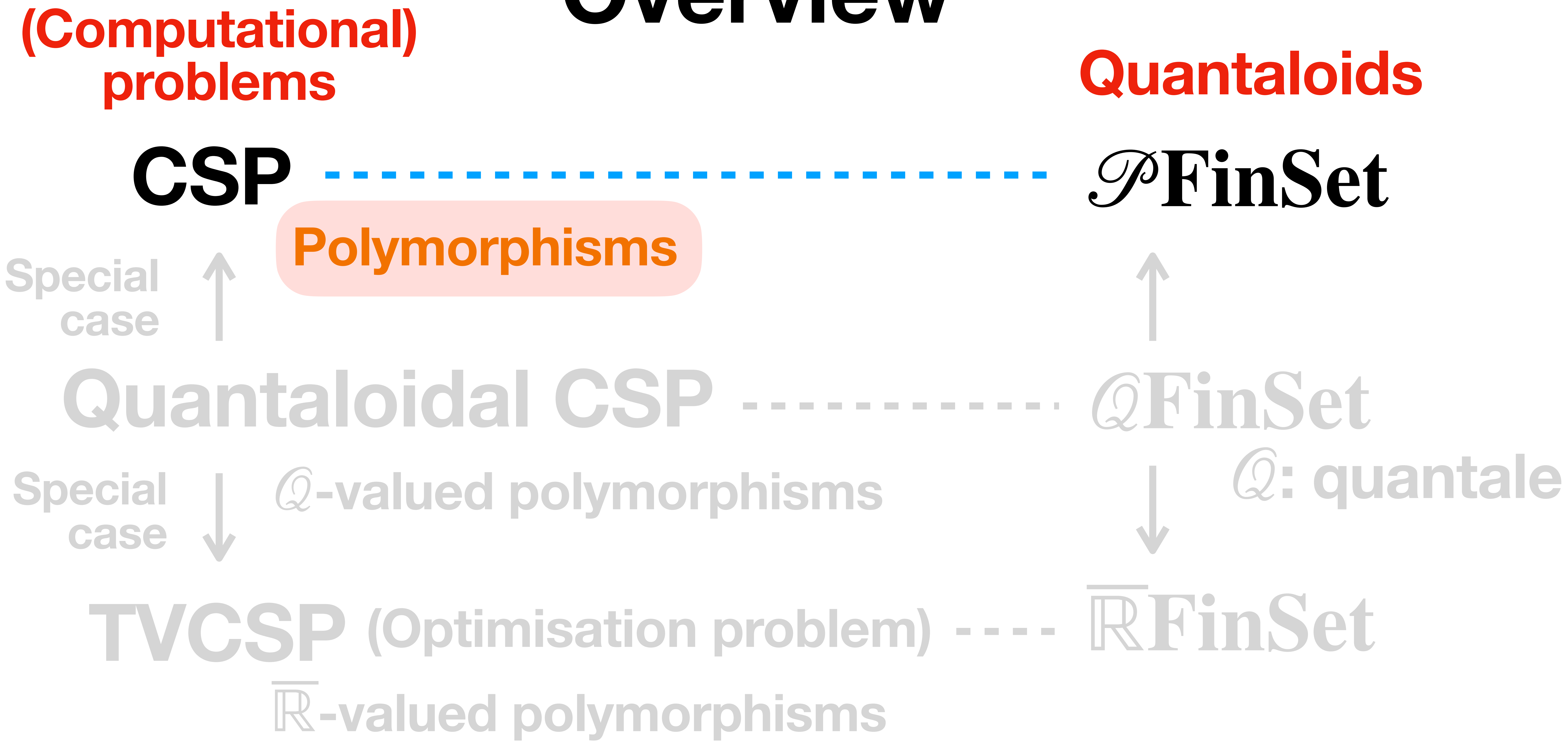
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Dichotomy theorem. [Bulatov 2017, Zhuk 2020]
For each “constraint language” $\mathscr{D}$,
$\mathrm{CSP}(\mathscr{D})$ is either in P or is NP-complete.

A constraint language $\mathscr{D}$ consists of

- **D: finite set**
- **$(\rho_i \subseteq D^{k_i})_{i \in I}$: finite family of relations on D.**

Finite relational structure

$\mathscr{D} = (D, (\rho_i)_{i \in I})$: constraint language
$\mathrm{CSP}(\mathscr{D})$: set of CSP instances defined by

$$I = (V, D', \mathscr{C}) \in \mathrm{CSP}(\mathscr{D}) \iff D' = D \text{ and } \forall (k, \mathbf{x}, \rho) \in \mathscr{C}, \rho \in \mathscr{D}$$

When is $\mathrm{CSP}(\mathscr{D})$ easy to solve?

- $\mathrm{CSP}(\mathscr{D})$ **is in P if $\mathscr{D}$ admits enough "symmetry"**
- $\mathrm{CSP}(\mathscr{D})$ **is NP-complete otherwise**

The relevant "symmetry" of $\mathscr{D}$ is captured by polymorphisms of $\mathscr{D}$

= homomorphisms (of relational structures) $\mathscr{D}^n \to \mathscr{D}$.

Dichotomy theorem. [Bulatov 2017, Zhuk 2020]

$\mathscr{D}$: constraint language $\quad \forall x, y, z \in D \,.\, f(y, x, y, z) = f(x, y, z, x)$

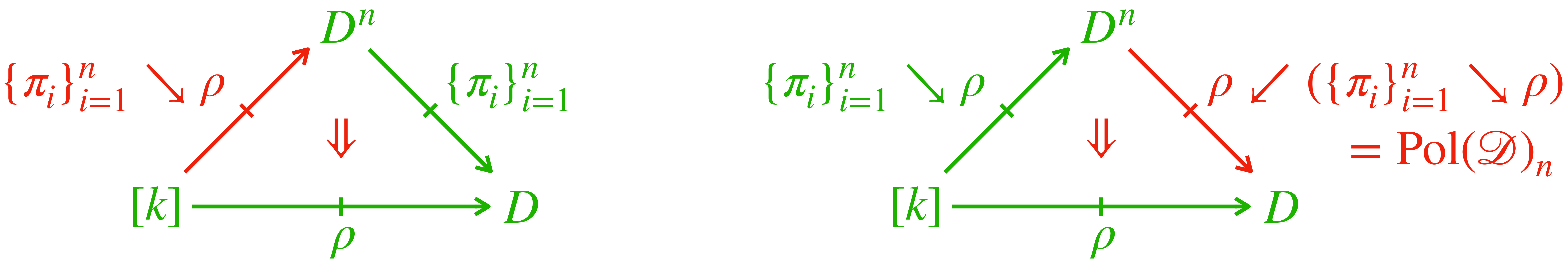
- $\mathrm{CSP}(\mathscr{D})$ is in P if $\mathscr{D}$ admits a Siggers operation $f\colon D^4 \to D$ as a polymorphism
- $\mathrm{CSP}(\mathscr{D})$ is NP-complete otherwise.

$\mathscr{D} = (D, (\rho_i)_{i \in I})$: **constraint language**

$\forall n \in \mathbb{N}$, **let** $\mathrm{Pol}(\mathscr{D})_n = \{n$**-ary polymorphisms of** $\mathscr{D}\}$
$= \{$**homomorphisms** $\mathscr{D}^n \to \mathscr{D}\}$

Assume I**: singleton, so that** $\mathscr{D} = (D, \rho \subseteq D^k)$**.**

Then $\mathrm{Pol}(\mathscr{D})_n : D^n \longrightarrow D$ **is given by:**



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Special case ↓

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A quantale is a one-object quantaloid.

Explicitly,

$\mathcal{Q} = (Q, \leq, e, \otimes)$ **is a quantale if**

- $(Q, \leq)$: **complete lattice**
- $(Q, e, \otimes)$: **monoid**

satisfying:

$$\alpha \otimes \left(\bigvee_{i \in I} \beta_i\right) = \bigvee_{i \in I} (\alpha \otimes \beta_i) \qquad \left(\bigvee_{i \in I} \alpha_i\right) \otimes \beta = \bigvee_{i \in I} (\alpha_i \otimes \beta)$$

$\mathcal{Q} = (Q, \leq, e, \otimes)$: quantale

The quantaloid $\mathcal{Q}\mathbf{FinSet}$:

Obj. Finite sets

Mor.

$$\frac{A \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} B}{\varphi : [A, B] \to Q}$$

Comp. $A \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} B \overset{\psi}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} C$

$$(\psi \circ \varphi)(h) = \bigvee \{\, \psi(g) \otimes \varphi(f) \mid f : A \to B, g : B \to C, g \circ f = h \,\}$$

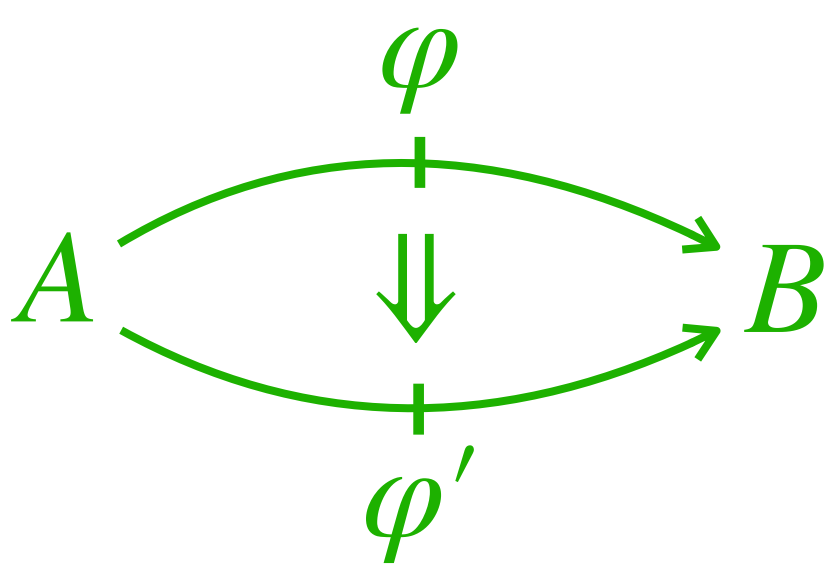
"Singleton" morphism

$$\frac{A \overset{f}{\longrightarrow} B}{\dfrac{A \overset{\{f\}}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} B}{\{f\} : [A, B] \to Q}}$$

$$g \longmapsto \begin{cases} e & \text{if } g = f \\ \bot_Q & \text{otherwise} \end{cases}$$

Id. $A \overset{\{\mathrm{id}_A\}}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} A$

2-cell



$$\overline{\overline{\varphi \leq \varphi'}}$$

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$\mathscr{Q} = 2$

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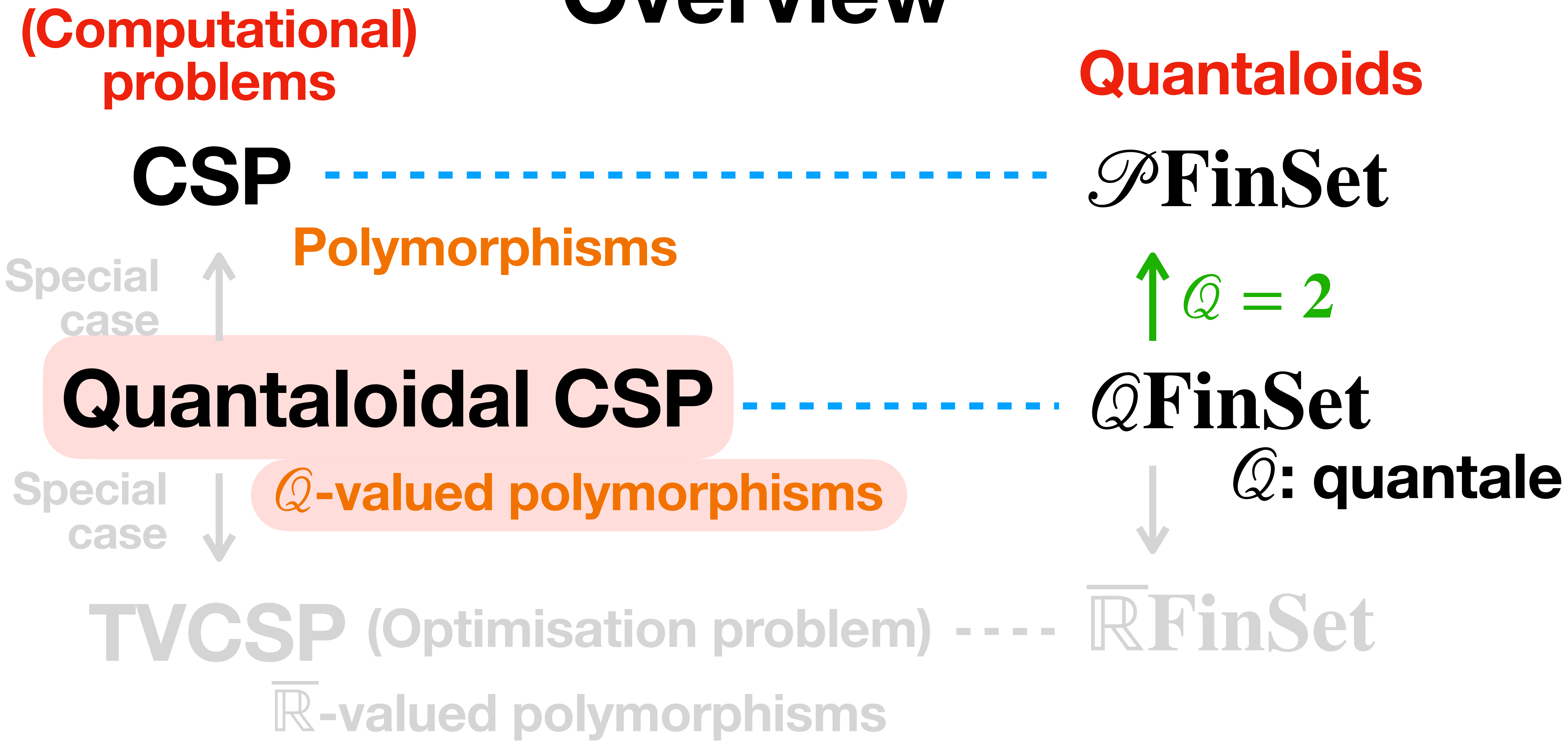

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$\mathcal{Q} = (Q, \leq, e, \otimes)$: quantale

A $\mathcal{Q}$-valued CSP instance $I = (V, D, \mathscr{C})$ consists of:

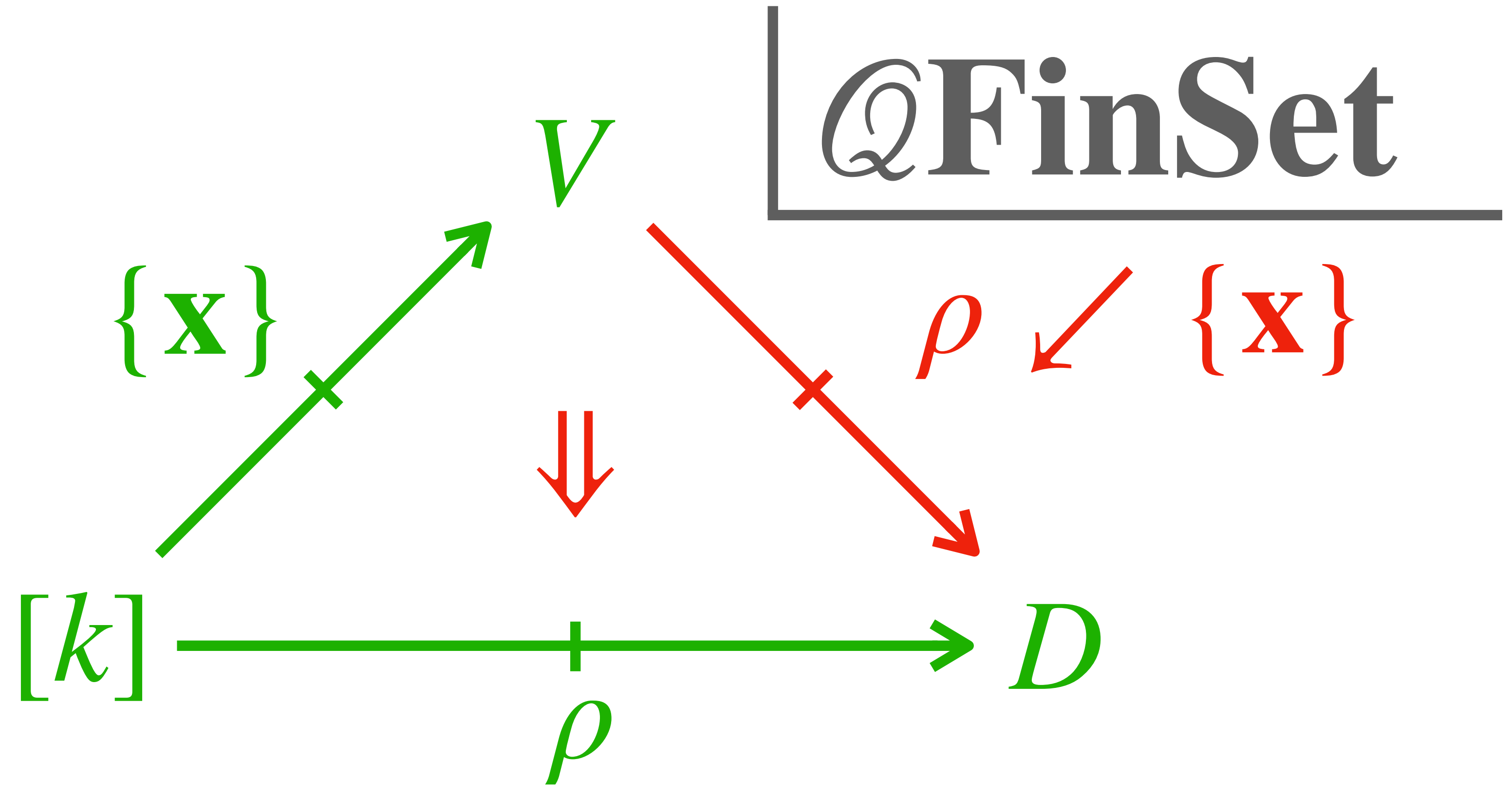
- V: finite set of variables
- D: finite set called the domain
- $\mathscr{C}$: finite set of "$\mathcal{Q}$-valued constraints"

A $\mathcal{Q}$-valued constraint is $(k, \mathbf{x}, \rho)$ where

- $k \in \mathbb{N}$, $\mathbf{x} \in V^k$, ~~$\rho \subseteq D^k$.~~

$$\frac{\rho\colon [k] \longrightarrow\!\!\!\!\!\!\!+\;\; D \text{ in } \mathcal{Q}\mathbf{FinSet}}{\rho\colon D^k \to Q}$$

Each $\mathcal{Q}$-valued constraint $(k, \mathbf{x}, \rho)$ yields



$$\frac{\mathcal{S}(I) = \bigwedge_{(k,\mathbf{x},\rho)\in\mathscr{C}} \rho \swarrow \{\mathbf{x}\}\colon V \longrightarrow\!\!\!\!\!\!\!+\;\; D}{\mathcal{S}(I)\colon [V, D] \to Q}$$

A $\mathcal{Q}$-valued **constraint language** $\mathcal{D}$ consists of

- D: finite set
- ~~$(\rho_i \subseteq D^{k_i})_{i \in I}$: finite family of relations on D.~~

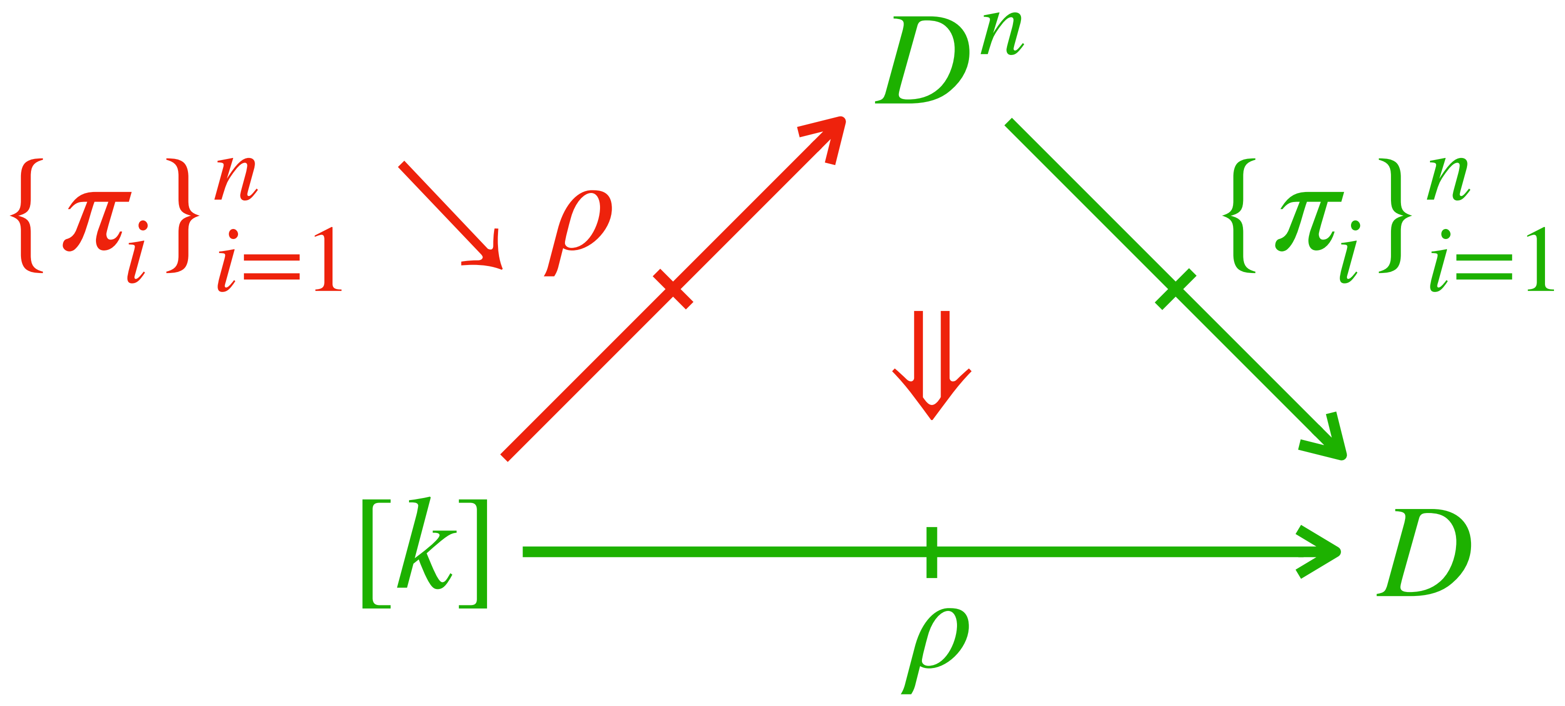
  $(\rho_i \colon [k_i] \longrightarrow\!\!\!\!\!\!+\ D)_{i \in I}$: finite family of morphisms in $\mathcal{Q}\mathbf{FinSet}$

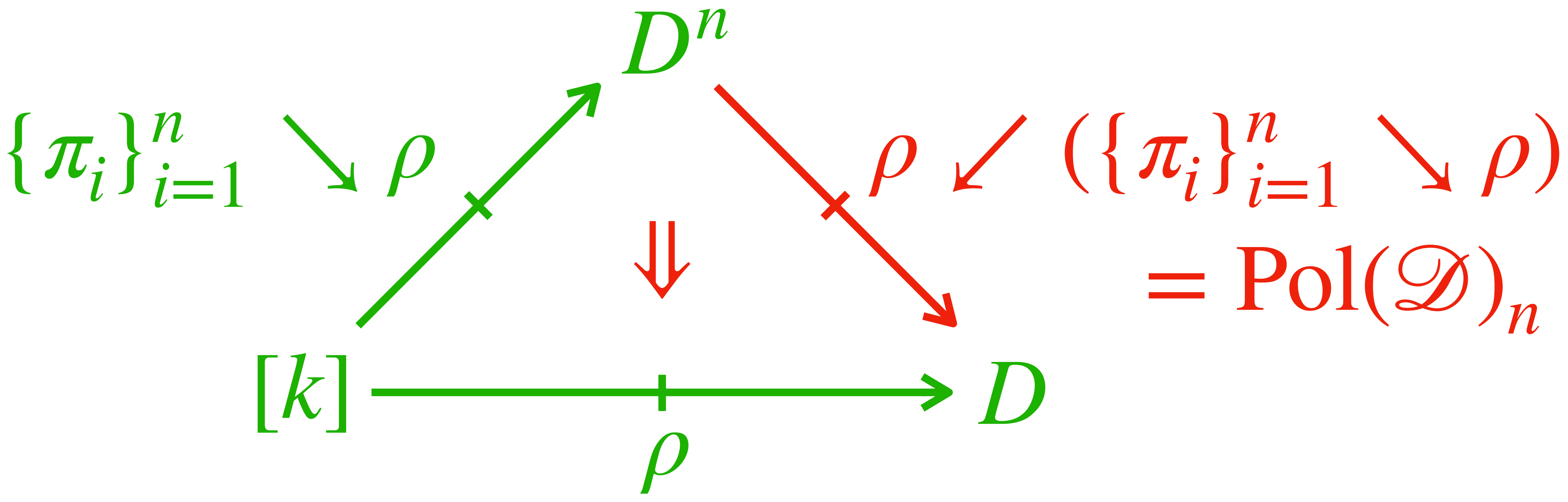
Assume I: singleton, so that $\mathcal{D} = (D, \rho \colon [k] \longrightarrow\!\!\!\!\!\!+\ D)$.

Then $\mathrm{Pol}(\mathcal{D})_n \colon D^n \longrightarrow\!\!\!\!\!\!+\ D$ is given by:

$$\mathrm{Pol}(\mathcal{D})_n \colon [D^n, D] \to Q$$

$\mathrm{Pol}(\mathcal{D})_n(f) \in Q$: the "degree" to which $f \colon D^n \to D$ is a polymorphism of $\mathcal{D}$





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Special case ↑   ↑ $\mathcal{Q} = 2$

Quantaloidal CSP ---------- $\mathcal{Q}\mathbf{FinSet}$

$\mathcal{Q}$: quantale

Special case ↓ $\mathcal{Q}$-valued polymorphisms   ↓ $\mathcal{Q} = \overline{\mathbb{R}}$

TVCSP (Optimisation problem) ---------- $\overline{\mathbb{R}}\mathbf{FinSet}$

$\overline{\mathbb{R}}$-valued polymorphisms

Letting $\mathcal{Q} = \overline{\mathbb{R}} = (\mathbb{R} \cup \{\pm\infty\}, \geq, 0, +)$ (cf. [Lawvere 1973]), we obtain a class of optimisation problems:

$$\inf_{s\colon V \to D} \sup_{(k,\mathbf{x},\rho)\in\mathscr{C}} \rho(s(x_1), \ldots, s(x_k))$$

which we call "tropical valued CSPs".

Dichotomy theorem for TVCSPs.*

$\mathscr{D}$: $\overline{\mathbb{R}}$-valued constraint language

- $\mathrm{TVCSP}(\mathscr{D})$ is in P if there exists a Siggers operation $f\colon D^4 \to D$ with $0 \geq \mathrm{Pol}(f)_4$.
- $\mathrm{TVCSP}(\mathscr{D})$ is NP-hard otherwise.

*** For a slightly more expressive version of TVCSPs.**

Summary

(Computational) problems | **Quantaloids**

CSP - - - - - $\mathscr{P}\mathbf{FinSet}$

Special case ↑ — Polymorphisms, Dichotomy theorem — ↑ $\mathcal{Q} = 2$

Quantaloidal CSP - - - - - $\mathcal{Q}\mathbf{FinSet}$

$\mathcal{Q}$: quantale

Special case ↓ — $\mathcal{Q}$-valued polymorphisms — ↓ $\mathcal{Q} = \overline{\mathbb{R}}$

TVCSP (Optimisation problem) - - - - - $\overline{\mathbb{R}}\mathbf{FinSet}$

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Dichotomy theorem