Network Sheaves Valued in Categories of Adjunctions & Their Laplacians

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Cellular Sheaf Theory

Theorem ([J. Curry]). Suppose $\mathcal{D}$ is a complete category, and $\boldsymbol{P}_X$ is the poset of face relations of a (say) simplicial complex X. Then, there is an equivalence of categories

$$Sh_{\mathcal{D}}\big(\mathrm{Alex}(\boldsymbol{P}_X)\big) \simeq \mathcal{D}^{\boldsymbol{P}_X}.$$

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Alexandrov topology on the poset $\boldsymbol{P}_X$

functor category

Cellular Sheaf Theory

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A **cellular sheaf** is a functor

$$\mathcal{F}: \boldsymbol{P}_X \to \mathcal{D}$$

where $\mathcal{D}$ is an arbitrary category.

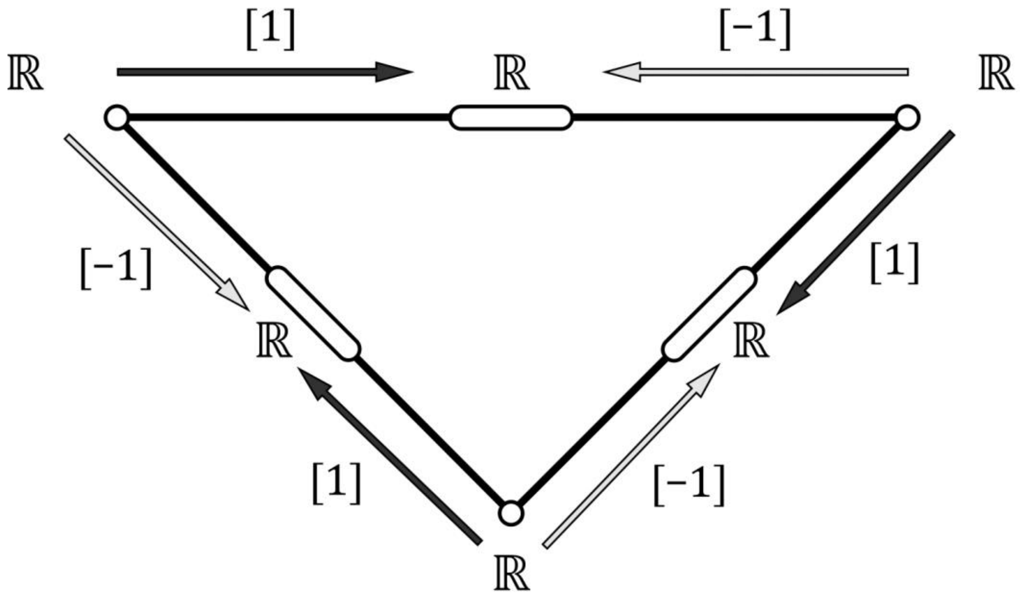
Cellular Sheaf Theory

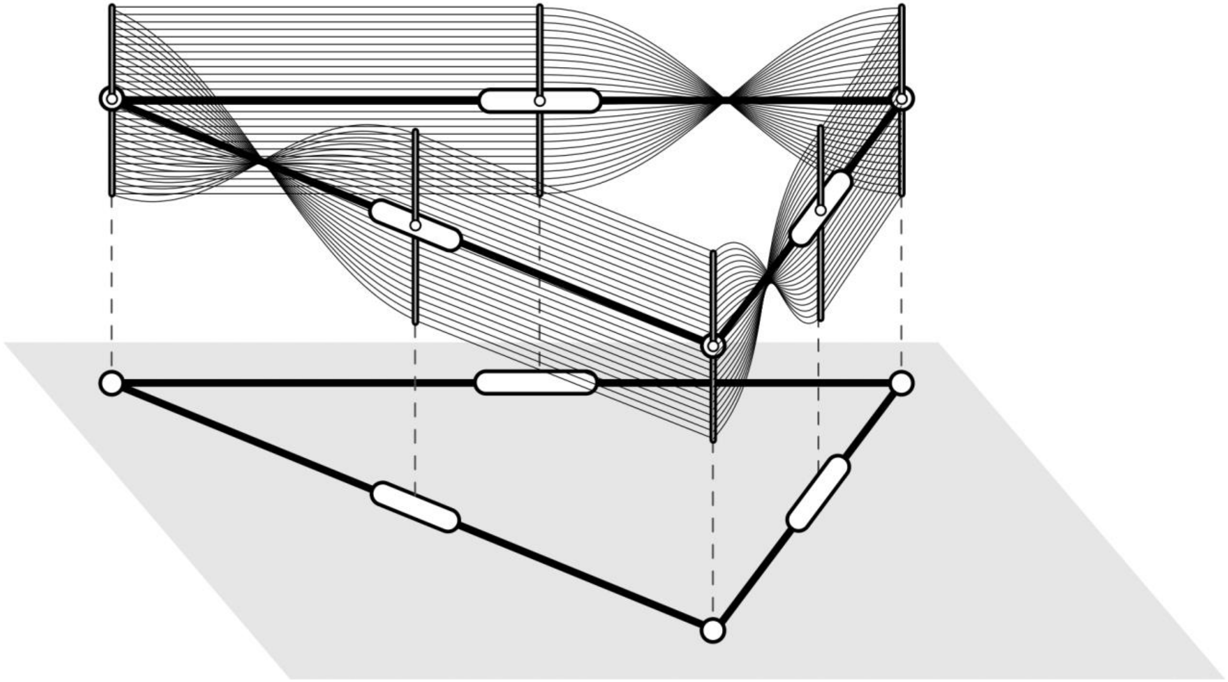
In this talk, we consider cellular sheaves where

- X is a graph, $G = (V_G, E_G)$
- $\mathcal{D}$ is a category of categories and adjunctions

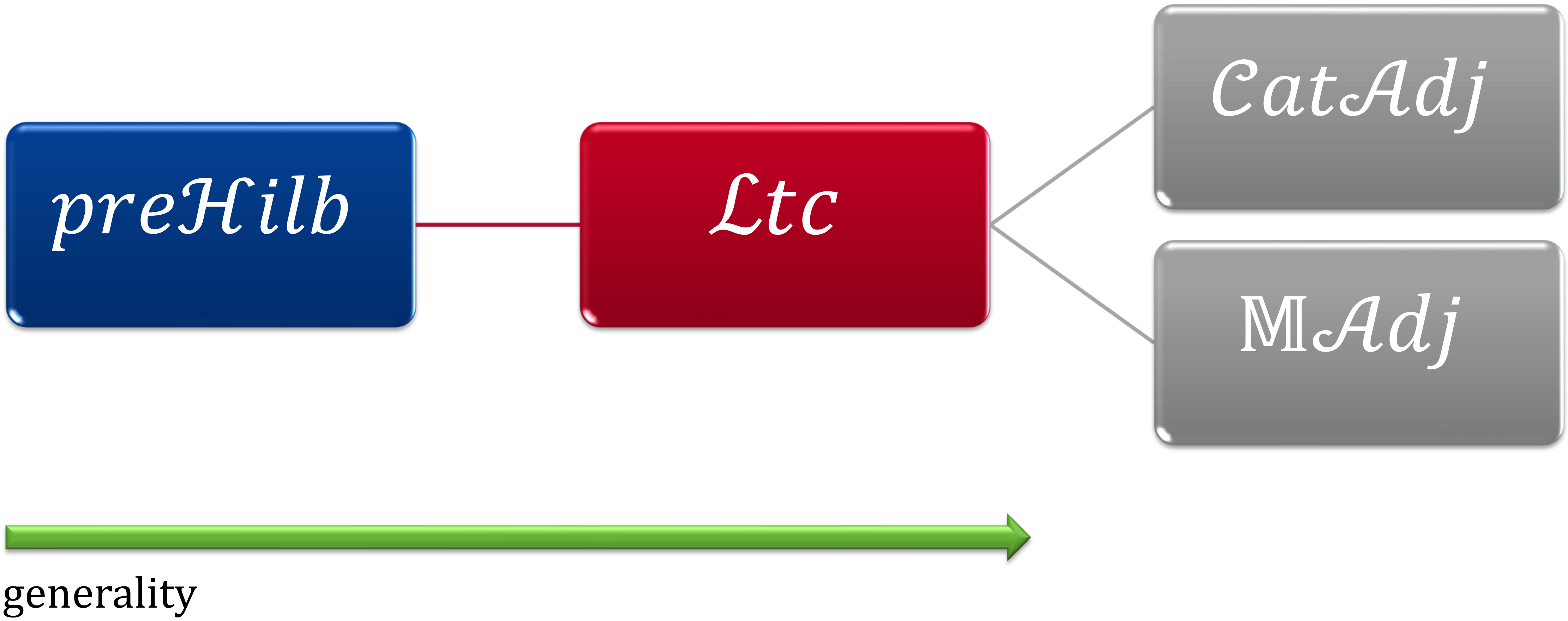
Our motivation is to compute limits i.e. **global sections**—consistent assignments of data to a sheaf.

Example

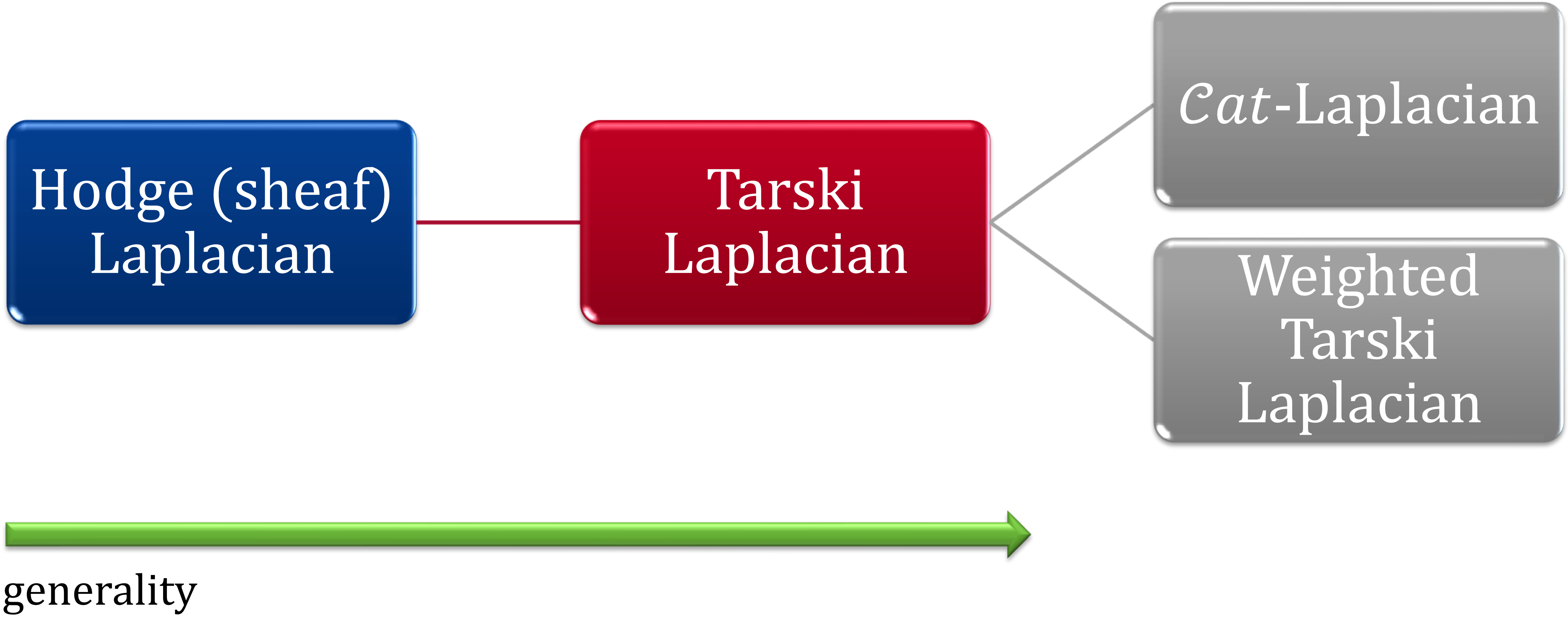


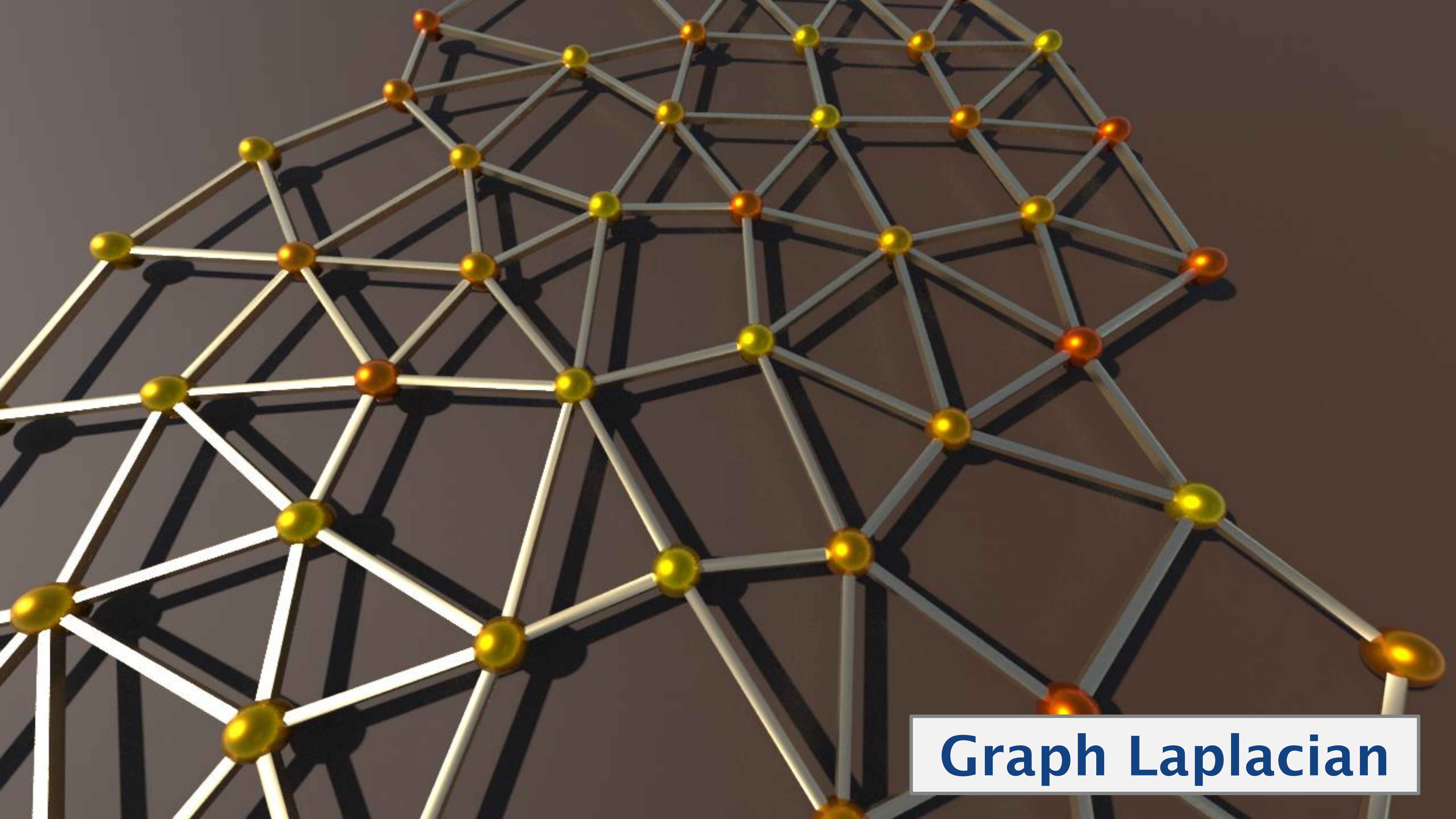


Categories of “Adjunctions”



Laplacians


Hodge (sheaf) Laplacian
Tarski Laplacian
$\mathcal{C}at$-Laplacian
Weighted Tarski Laplacian
generality



Graph Laplacian

Graph Laplacian

$G = (V_G, E_G, W_G)$ is a weighted graph

w_{ij} is weight of $ij \in E_G$

d_i is degree of $i \in V_G$
$n = |V_G|$

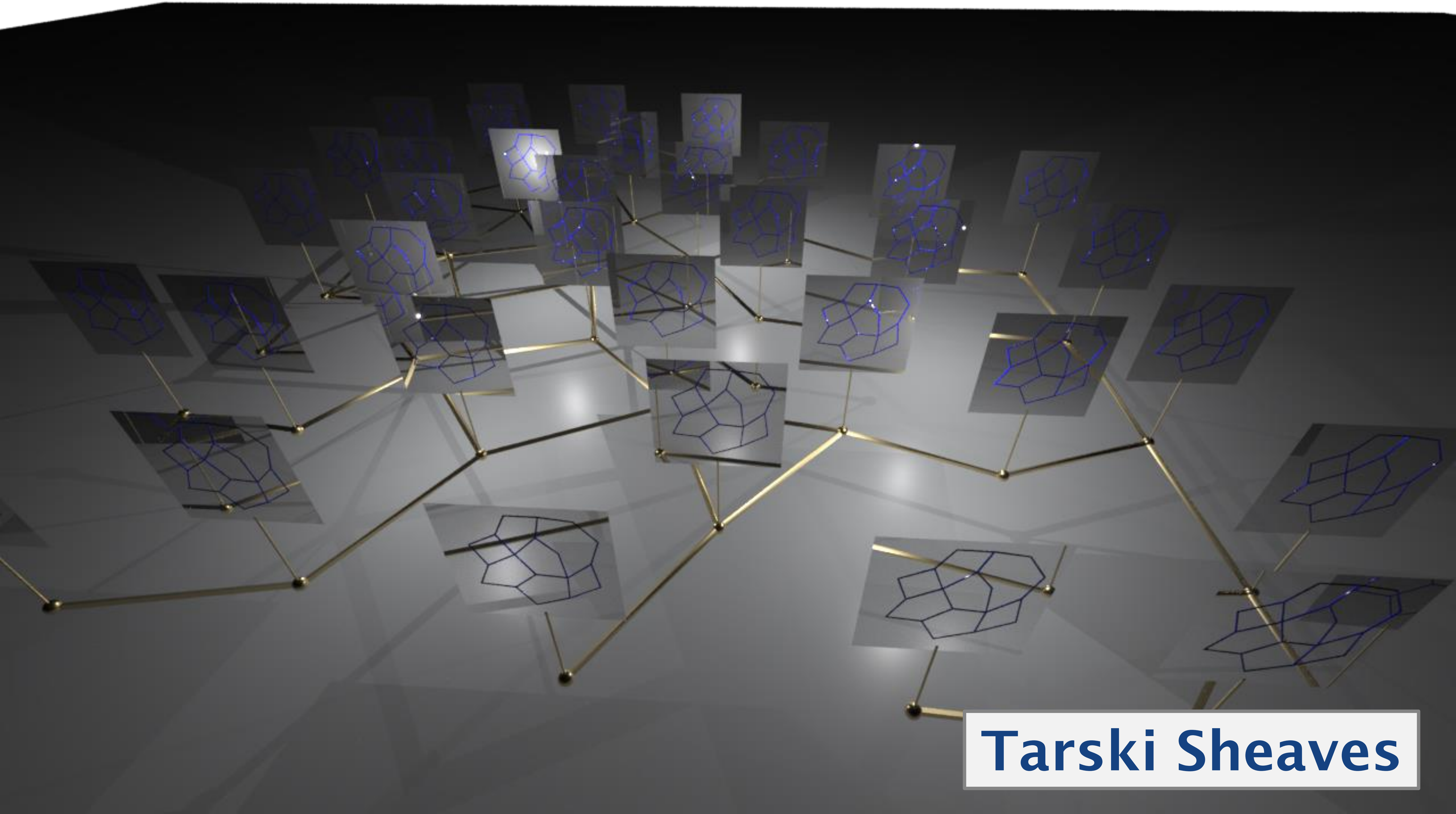
The **graph Laplacian** is a $n \times n$ matrix,

$$[L]_{ij} = \begin{cases} -w_{ij}, & ij \in E_G \\ d_i, & i = j \\ 0, & \text{else} \end{cases}$$

$\dim(\ker L) = \#$ connected components

$\dot{\boldsymbol{x}} = -L\boldsymbol{x}$ where $\boldsymbol{x}: V_G \to \mathbb{R}$ exponentially stable

Tarski Sheaves



Tarski Laplacian

$G = (V_G, E_G)$, a graph.

$\mathcal{F}: \boldsymbol{P}_G \to \mathcal{L}tc$, a functor.

Can we compute

$$\lim \left(\underline{\mathcal{F}}: \boldsymbol{P}_G \to \mathcal{S}up\right)$$

where $\underline{\mathcal{F}}: \boldsymbol{P}_G \to \mathcal{S}up$ forgets right adjoints?

Definition. The Tarski Laplacian is an order preserving map

$$\prod_{v \in V_G} \mathcal{F}(v) \xrightarrow{L} \prod_{v \in V_G} \mathcal{F}(v)$$

$$(L\boldsymbol{x})_v = \bigwedge_{e \in \delta v} \mathcal{F}^R_{v<e}\left(\bigwedge_{w \in \partial e} \mathcal{F}^L_{w<e}(x_w)\right)$$

A Fixpoint Theorem

Theorem. Let $\mathcal{F}: \boldsymbol{P}_G \to \mathcal{L}tc$ be a Tarski sheaf over G. Then,

$$Post(L) = \lim \underline{\mathcal{F}}$$

$$Post(L) = \{\boldsymbol{x}: L(\boldsymbol{x}) \succcurlyeq \boldsymbol{x}\}$$

$$\lim \underline{\mathcal{F}} = \left\{\boldsymbol{x} \in \prod_{v \in V_G} \mathcal{F}(v) : \mathcal{F}^L_{v<e}(x_v) = \mathcal{F}^L_{w<e}(x_w)\right\}$$

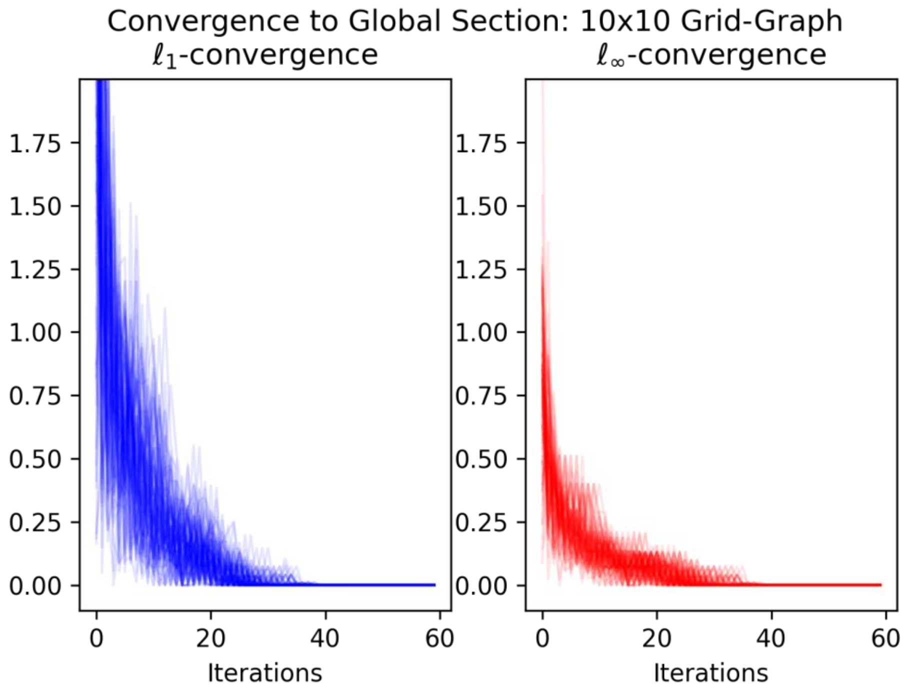
Mimics **Hodge Theorem**: $H^k(C^\cdot) \cong \ker L_k$

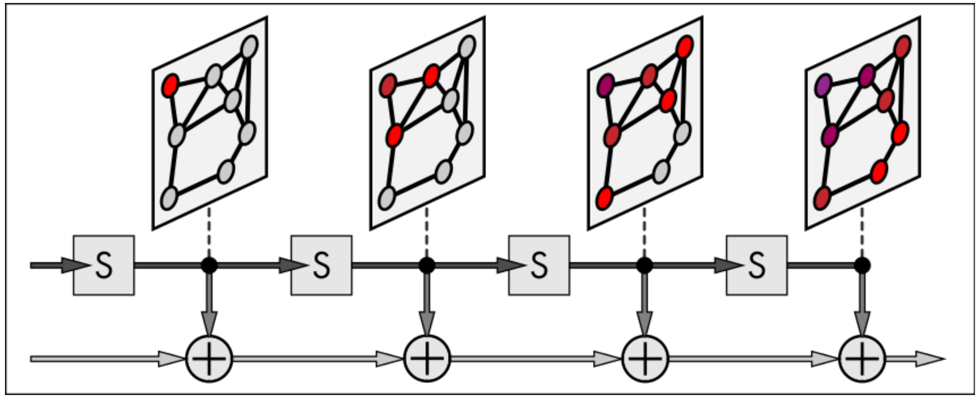
Corollary. $\lim \underline{\mathcal{F}}$ is a complete lattice.

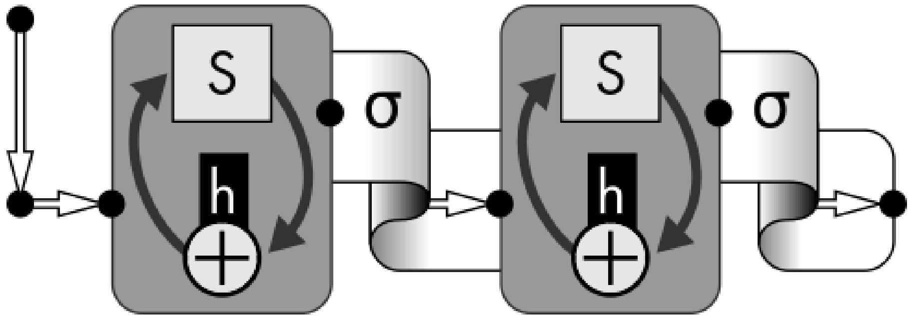
Proof. Tarski Fixed Point Theorem.

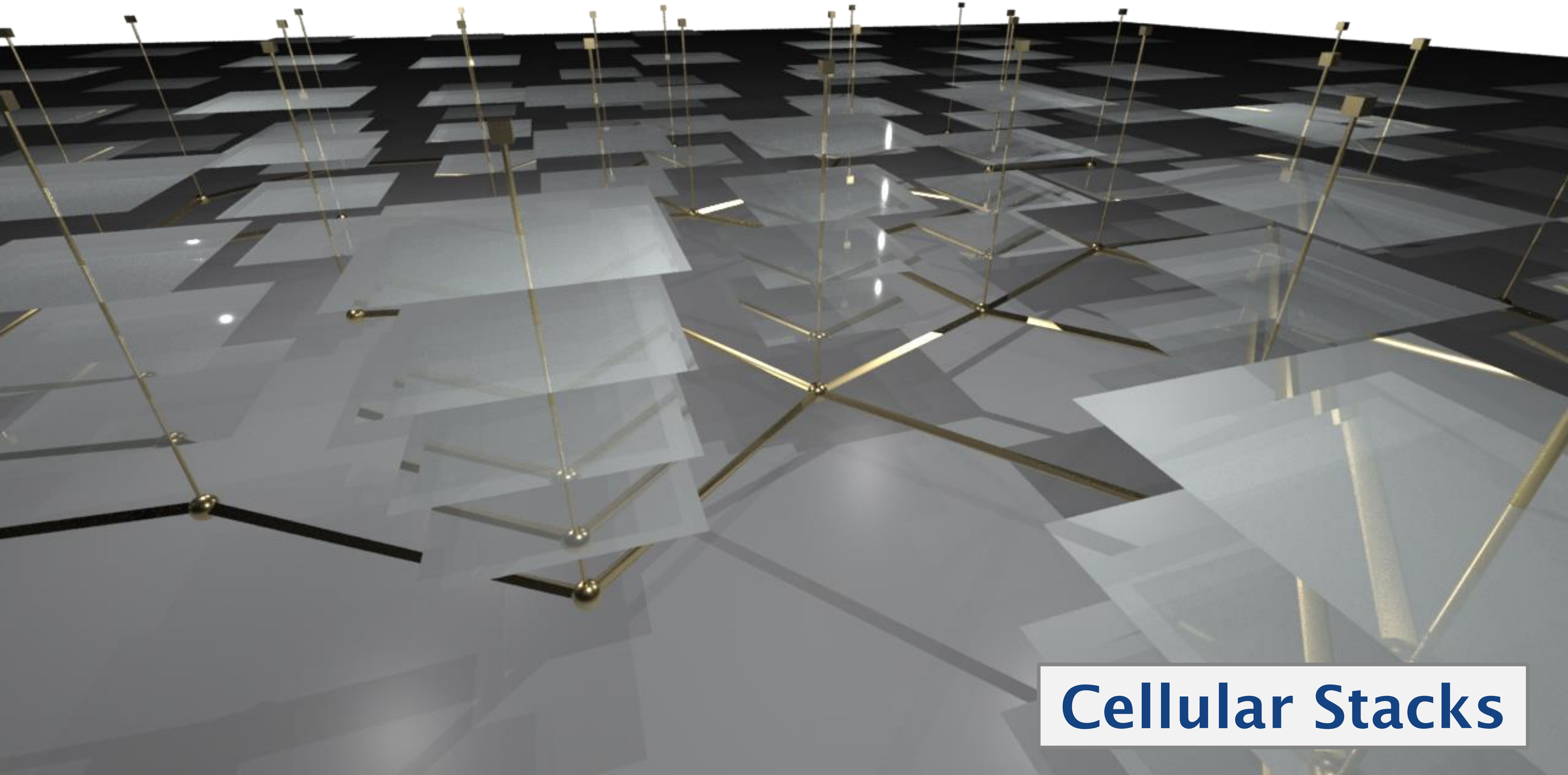
Toward Applications

- Graph Signal Processing (GSP)
- Formal Concepts
- Consensus









Cellular Stacks

$\mathcal{C}at$-Laplacian

$\mathcal{C}at\mathcal{A}dj$ is a 2-category.

$\mathcal{F}: \boldsymbol{P}_G \to \mathcal{C}at\mathcal{A}dj$, a **cellular adjunction stack**

Definition. The $\mathcal{C}at$-Laplacian is a functor

$$\prod_{v \in V_G} \mathcal{F}(v) \xrightarrow{L} \prod_{v \in V_G} \mathcal{F}(v)$$

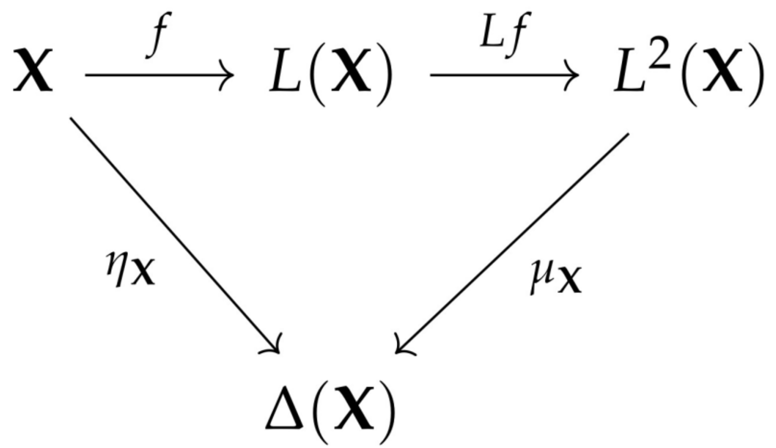
$$(L\boldsymbol{X})_v = \prod_{e \in \delta v} \prod_{w \in \partial e} \mathcal{F}^R_{v<e} \mathcal{F}^L_{w<e}(X_w)$$

A 2-Categorical Fixpoint Theorem

$$\Delta(\boldsymbol{X})_v = \prod_{e \in \delta v} \mathcal{F}^R_{v<e} \mathcal{F}^L_{v<e}(X_v)$$

$\eta : 1 \Rightarrow \Delta$

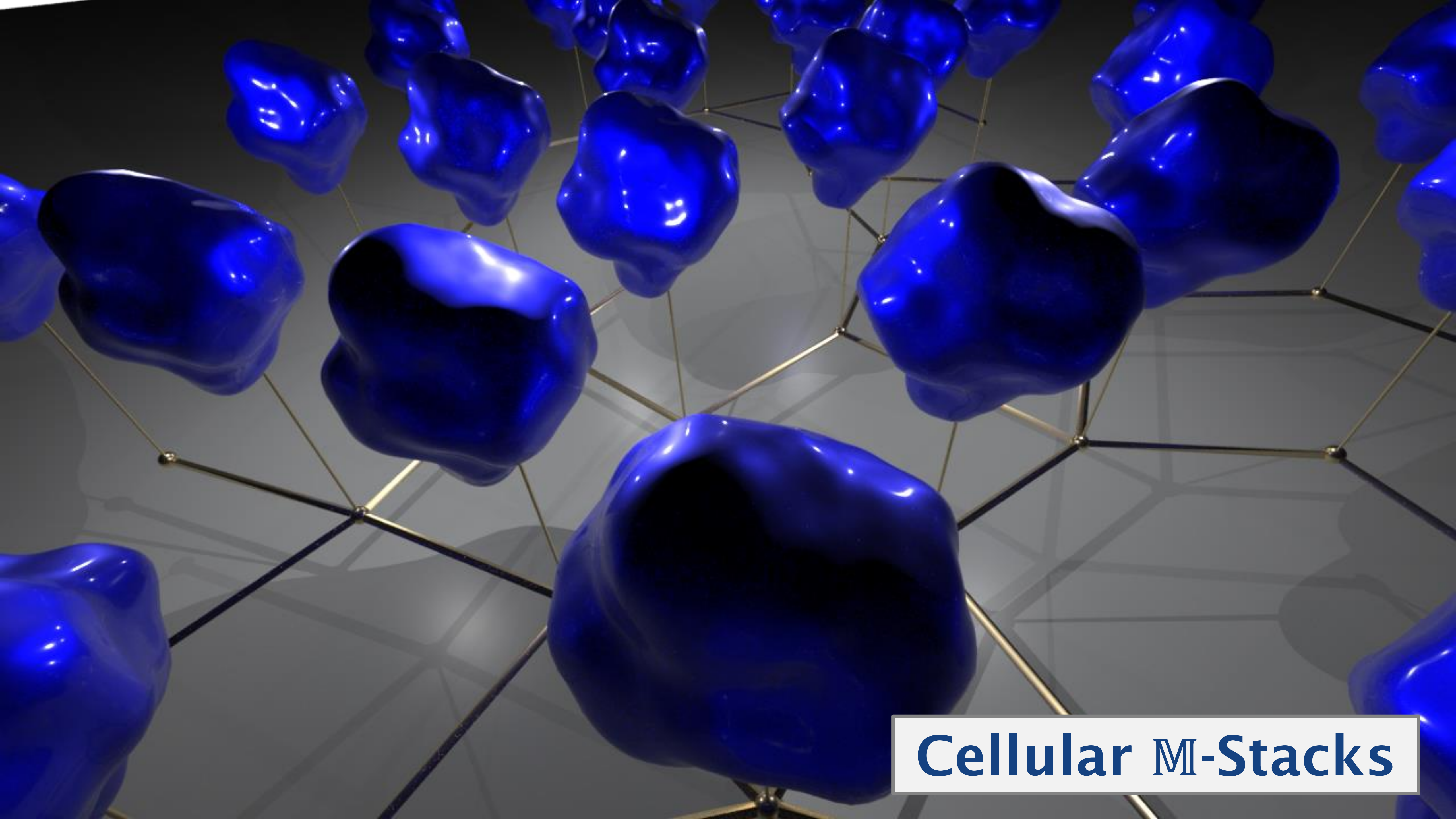
$\mu : L^2 \Rightarrow \Delta$



$Post(L) = \{\boldsymbol{X}, f : \boldsymbol{X} \to L(\boldsymbol{X}) : \mu_{\boldsymbol{X}} \circ Lf \circ f = \eta_{\boldsymbol{X}}\}$

$\underline{\mathcal{F}} :\ P_{\boldsymbol{G}} \to \mathcal{C}at$, a cellular stack

Theorem. $\lim \underline{\mathcal{F}} \simeq Post(L)$


Cellular $\mathbb{M}$-Stacks

$\mathbb{M}$-Categories

$\mathbb{M}$, closed monoidal thin (preorder) category whose unit is terminal.

e.g.

- $\mathbb{B}$, 2-element Boolean
- $\mathbb{I} = ([0,1], \cdot, 1, \leq)$
- $\mathbb{L} = ([0, \infty], +, 0, \geq)$
- $\mathbb{H}$, Heyting algebra

$\mathbb{M}\mathcal{A}dj$, category of $\mathbb{M}$-enriched and $\mathbb{M}$-adjunctions

Weighted Tarski Laplacian

$\mathcal{F}: \boldsymbol{P}_G \to \mathbb{M}\mathcal{C}at$, adjunction $\mathbb{M}$-stack

$\mathcal{W}: \boldsymbol{P}_G \to \mathbb{M}$, a weighting

Definition. The weighted Laplacian is a $\mathbb{M}$-functor

$$\prod_{v \in V_G} \mathcal{F}(v) \xrightarrow{L} \prod_{v \in V_G} \mathcal{F}(v)$$

$$(L\boldsymbol{X})_v = \prod_{\substack{e \in \delta v \\ w \in \partial e}}^{\mathcal{W}} \mathcal{F}^R_{v<e} \mathcal{F}^L_{w<e}(X_w)$$

An $\mathbb{M}$-Enriched Fixpoint Theorem

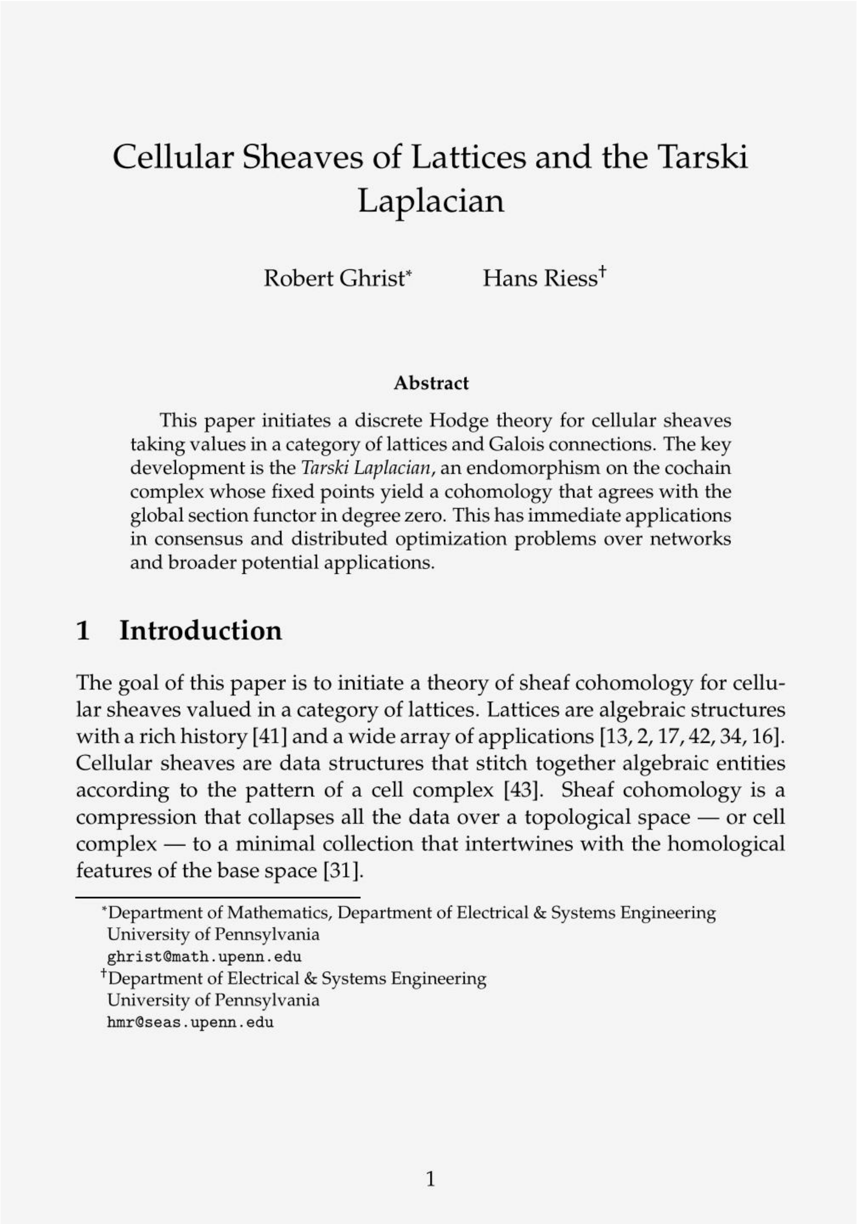
$\mathcal{F}: \boldsymbol{P}_G \to \mathbb{M}\mathcal{C}at$, adjunction $\mathbb{M}$-stack
$\mathcal{W}: \boldsymbol{P}_G \to \mathbb{M}$, a weighting
$m \in \mathbb{M}$, choice of object

Theorem.

$$\hom\big(\boldsymbol{X}, L(\boldsymbol{X})\big) \succcurlyeq m$$

if and only if $\forall\ v < e > w$

$$\Big[\mathcal{W}(e), \hom\Big(\mathcal{F}^L_{w<e}(X_w), \mathcal{F}^L_{v<e}(X_v)\Big)\Big] \succcurlyeq m$$



Cellular Sheaves of Lattices and the Tarski Laplacian

Robert Ghrist[*] Hans Riess[†]

Abstract

This paper initiates a discrete Hodge theory for cellular sheaves taking values in a category of lattices and Galois connections. The key development is the *Tarski Laplacian*, an endomorphism on the cochain complex whose fixed points yield a cohomology that agrees with the global section functor in degree zero. This has immediate applications in consensus and distributed optimization problems over networks and broader potential applications.

1 Introduction

The goal of this paper is to initiate a theory of sheaf cohomology for cellular sheaves valued in a category of lattices. Lattices are algebraic structures with a rich history [41] and a wide array of applications [13, 2, 17, 42, 34, 16]. Cellular sheaves are data structures that stitch together algebraic entities according to the pattern of a cell complex [43]. Sheaf cohomology is a compression that collapses all the data over a topological space — or cell complex — to a minimal collection that intertwines with the homological features of the base space [31].

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- our paper (on Tarski Laplacian): to appear in *Homology, Homotopy, and Applications*:
- accessible: arxiv.org/abs/2007.04099
- preprint w/ **Paige North** on $\mathbb{M}$-stacks: TBD
- you can follow me on twitter: @hansmriess
- you can email me: hmr@seas.upenn.edu
- my website: www.hansriess.com

Thank you.