Applied Category Theory 2021

TEMPORAL LANDSCAPES: A GRAPHICAL LOGIC OF BEHAVIOR

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MOTIVATION

- The work presented here inserts itself within the broader need of *modeling* and *analyzing* complex systems of systems

- The starting point was the need to reinterpret the classical notions developed in engineering that are taught in a course like *Signals & Systems*

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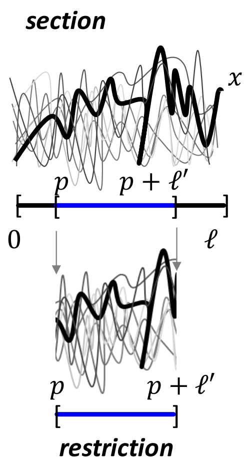
Signals



Sheaves over times intervals

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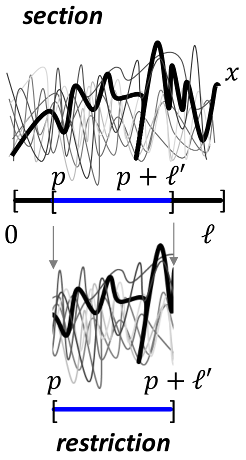

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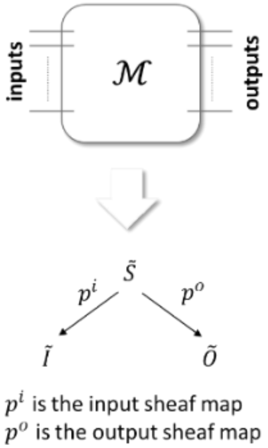

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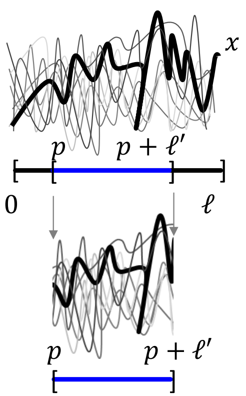

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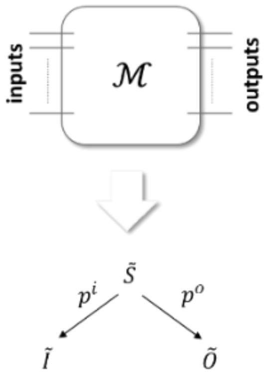
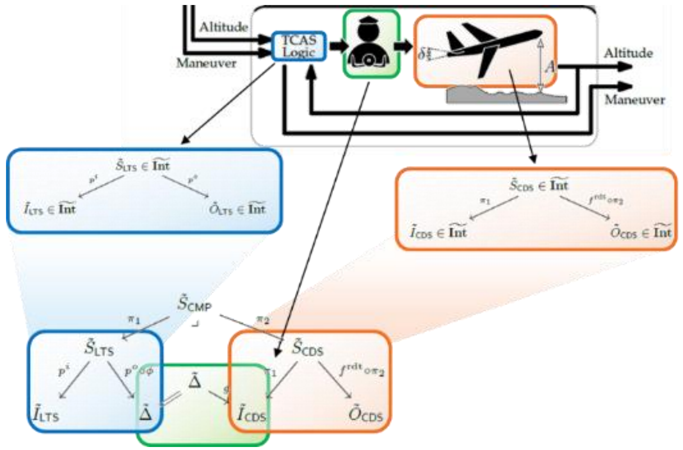


p^i is the input sheaf map
p^o is the output sheaf map

System of Systems



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  - We can <u>model</u> the behavior of the overall system of systems using composition of abstract machines and describe it as a sheaf over a time interval

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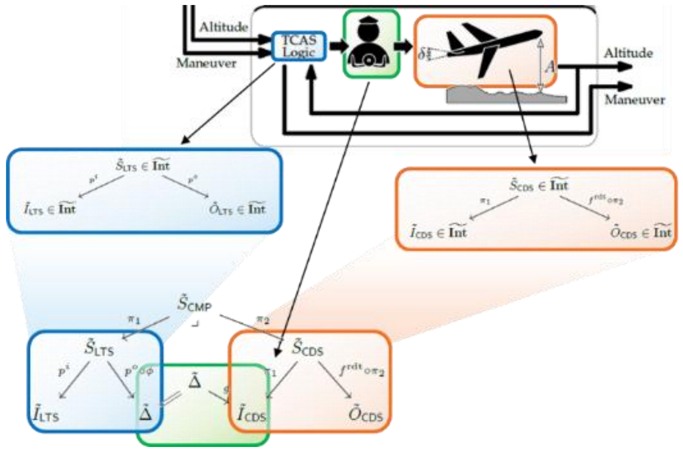

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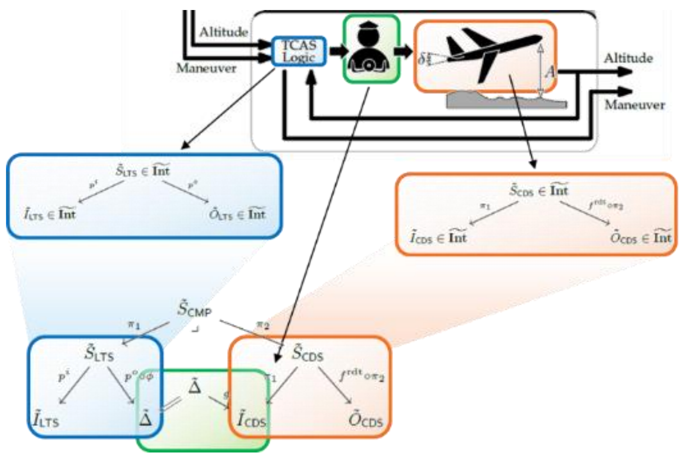
  - How do we analyze the system's behavior in a way that is mathematically sound but also **easy/intuitive for non-expert**?

System of Systems



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- **Challenge:**
  - We can model the behavior of the overall system of systems using composition of abstract machines and describe it as a sheaf over a time interval

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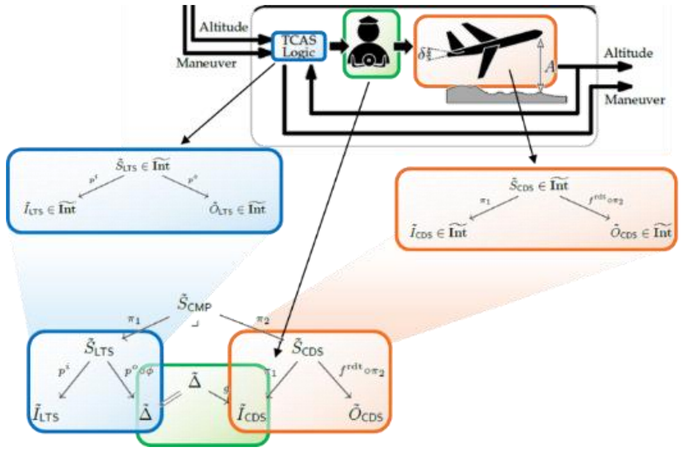
  - How do we analyze the system's behavior in a way that is m **TOPOS THEORY** nd but also e **TEMPORAL** e

LANDSCAPES ?

System of Systems



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TEMPORAL LANDSCAPES

- Temporal landscapes provide the truth values of a logical system which we call ***temporal landscape logic***

Definition: *A* temporal landscape *on* $\mathbb{R}$ *is a set* L *of time intervals* $[t_1, t_2] \subseteq \mathbb{R}$, $t_1 \leq t_2$ *such that:*

(a) [Down-closure] If $[t_1, t_2] \in L$ *and* $t_1 \leq t_1' \leq t_2' \leq t_2$ *then* $[t_1', t_2'] \in L$

(b) [Openness] If $[t_1, t_2] \in L$ *then there exists* $t_1' < t_1 \leq t_2 \leq t_2'$ *such that* $[t_1', t_2'] \in L$

For the theoretical foundations of the temporal type theory, see P. Schultz and D.I. Spivak. "Temporal Type Theory: A topos-theoretic approach to systems and behavior". Springer, Birkhäuser, 2019

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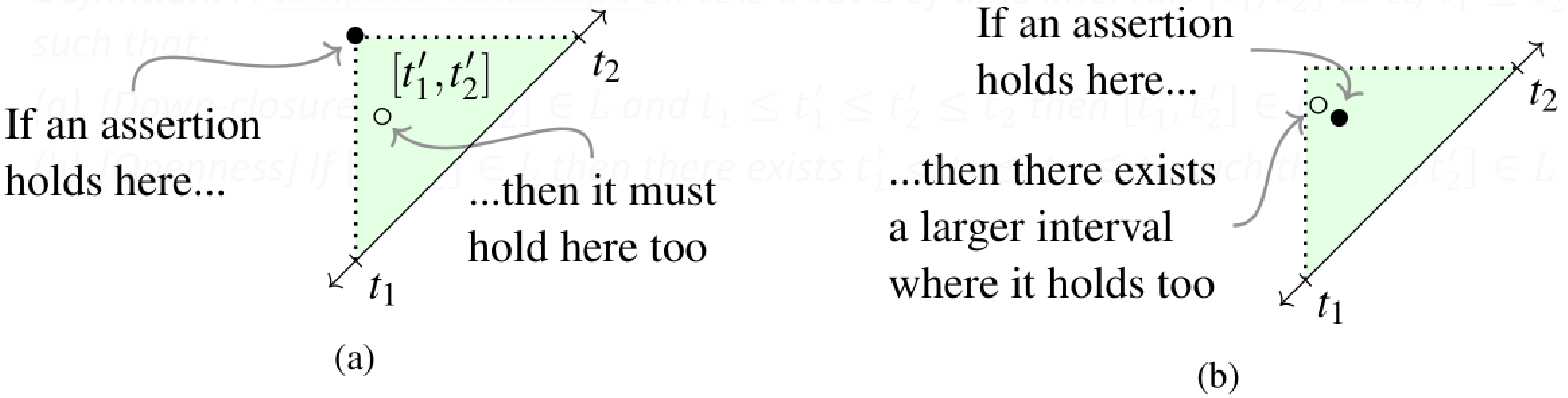

- The name “temporal landscape” comes from its resemblance to the “persistence landscapes” used in TDA

TEMPORAL LANDSCAPES

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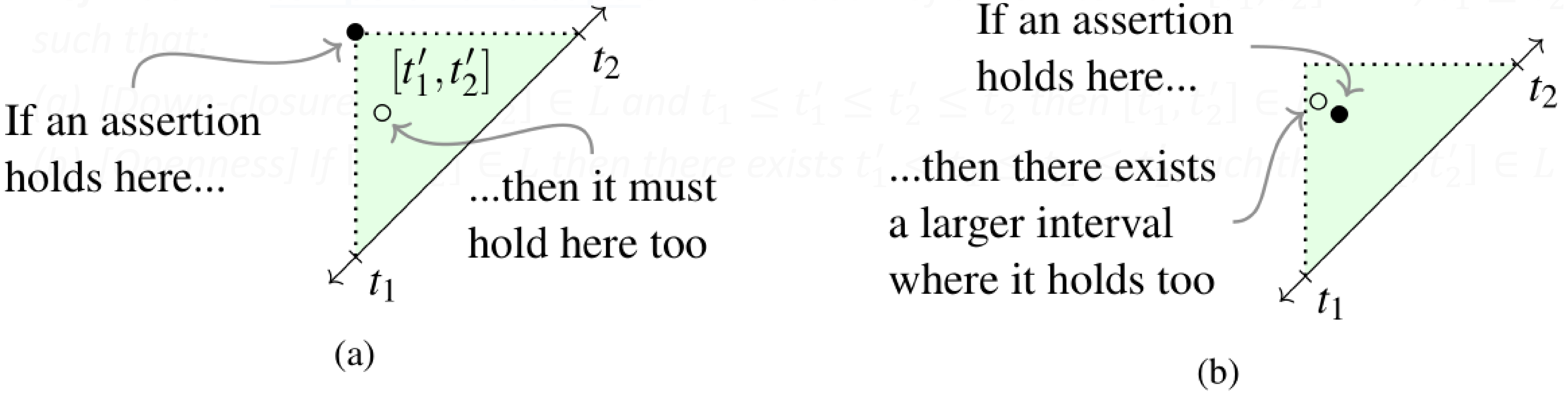

- We write Prop for the set of temporal landscapes, namely $\mathrm{Prop} = \mathrm{Hom}(1, \Omega)$

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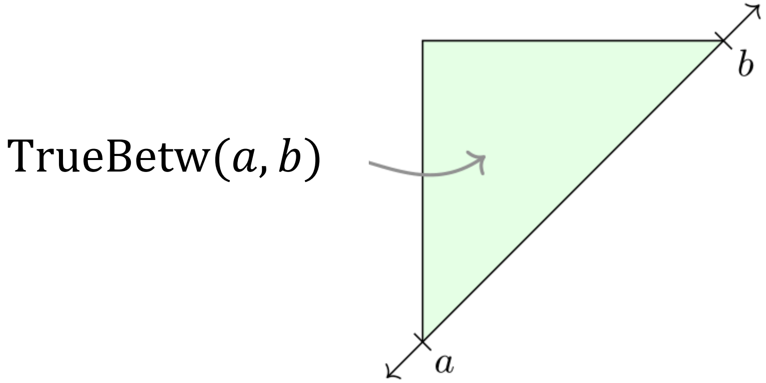
"ROOF", ALWAYS-TRUE AND ALWAYS-FALSE

- Given a pair $a < b$ in $\mathbb{R}$ the **roof** over a, b is the temporal landscape:

$$\text{TrueBetw}(a, b) \coloneqq \{[t_1, t_2] | a < t_1 \leq t_2 < b\}$$



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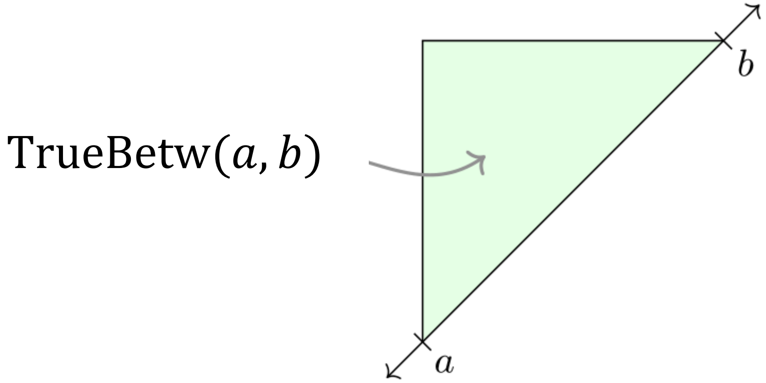
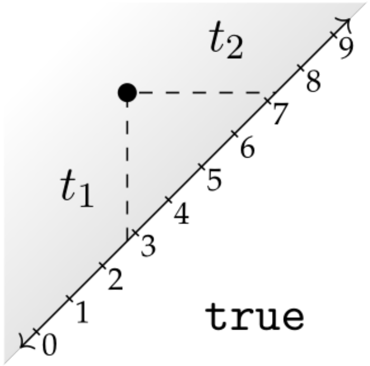


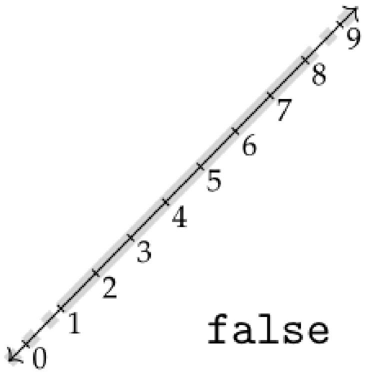
- Given a pair $a < b$ in $\mathbb{R}$ the **roof** over a, b is the temporal landscape:

$$\mathrm{TimeBetw}(a, b) \coloneqq \{[t_1, t_2] | a < t_1 \leq t_2 < b\}$$



- The landscape for True is the maximal landscape:

$$\mathrm{true} \coloneqq \{[t_1, t_2] \mid t_1 < t_2 \in \mathbb{R}\}$$



- The landscape for False is the minimal landscape, containing no intervals:

$$\mathrm{false} \;\coloneqq \emptyset$$

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AND, OR, IMPLICATION

- The conjunction (disjunction) of two landscapes, φ and ψ are just their intersection (union)
- Given the temporal landscapes φ and ψ we have:

$$(\varphi \Rightarrow \psi) \coloneqq \{[a, b] \mid \mathrm{TrueBetw}(a, b) \cap \varphi \subseteq \psi\}$$

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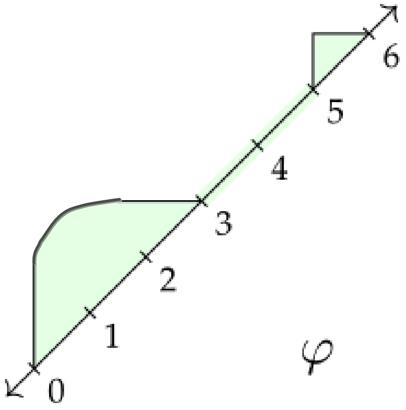
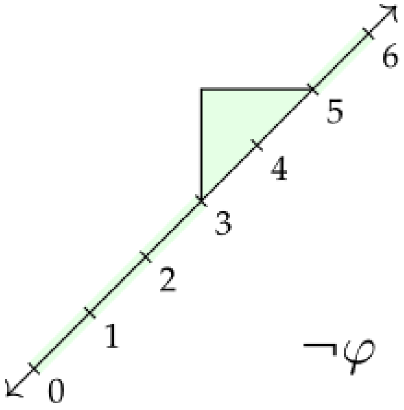
This means that given φ the landscape of the negation is obtained by drawing a roof in φ whenever the landscape is false (flat on the diagonal)





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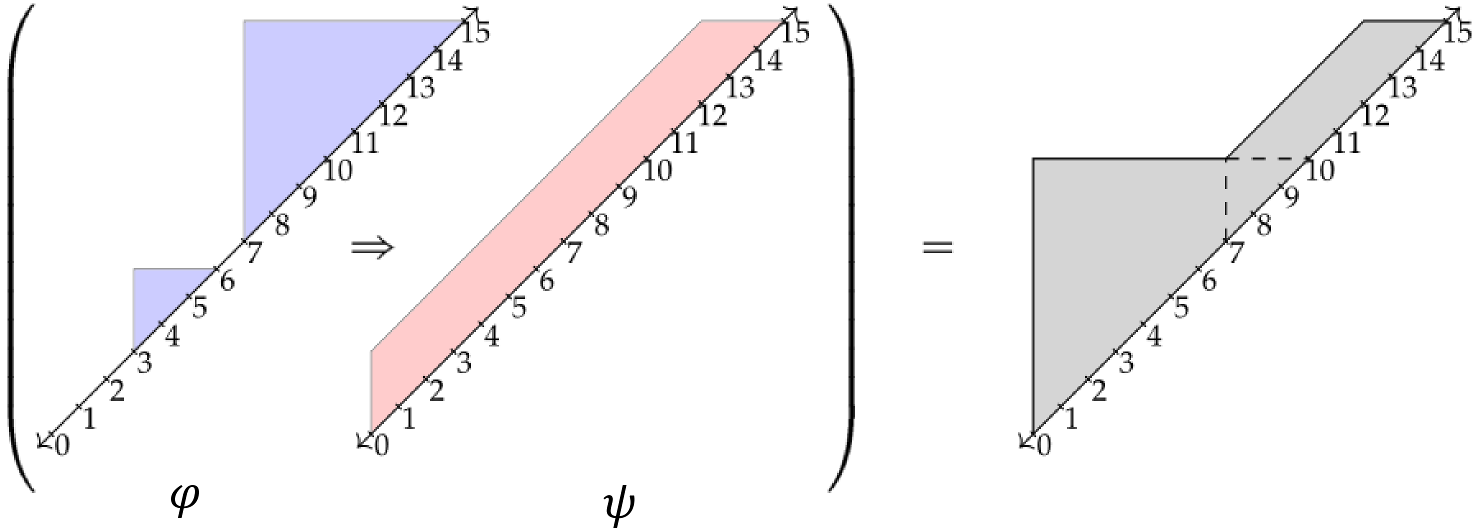
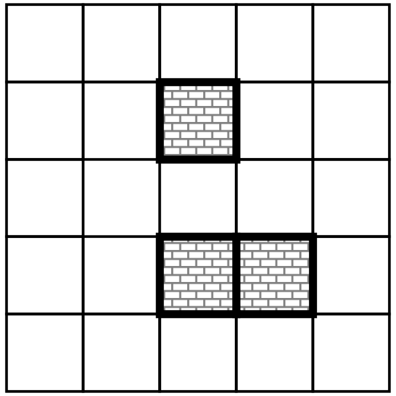
- The visual intuition of the implication generalizes that of the negation in that $\varphi \Rightarrow \psi$ contains a roof over all time intervals within which φ is contained in ψ:



EXAMPLE IN A STATIC GRID WORLD



- Consider an environment modelled as a grid (V, E) that *does not change over time*.
- Then we take the vertices V and construct a *constant behavior type* V and then take E to be the *constant subtype* of $V \times V$ consisting of pairs of vertices.

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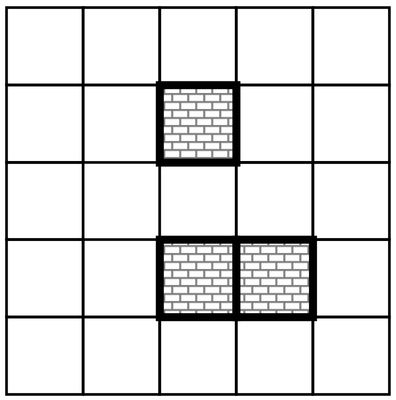

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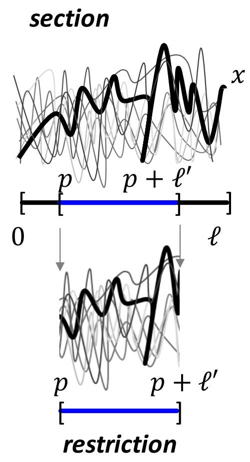

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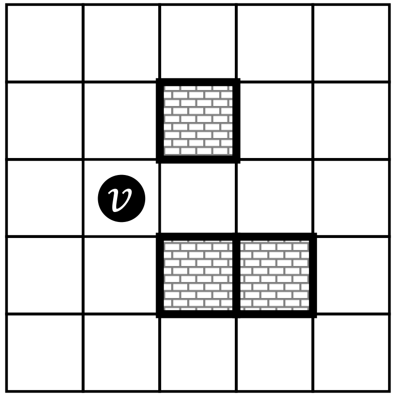

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- Then we take the vertices V and construct a *constant behavior type* V and then take E to be the *constant subtype* of $V \times V$ consisting of pairs of vertices.
- The constancy of the subtype E really models the fact that the adjacency relation does not change over time:

$$\forall(v, v' : V).\, E(v, v') \vee \neg E(v, v')$$

  where we consider $E : V \times V \to \mathrm{Prop}$

- Given such a predicate we can ask trivial questions such as if two vertices are adjacent or not or if the agent in position v is adjacent to a cell that contains a wall or if an agent is in between two walls, obtaining *always-true* or *always-false* temporal landscapes

INTUITIONISTIC LOGIC

- Let us consider the predicate

$$Occ(v): V \to \mathrm{Prop}$$

  capturing the fact that a cell can be occupied

- The temporal landscape $Occ(v)$ represents the time intervals a cell v is occupied

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- The temporal landscape $Occ(v)$ represents the time intervals a cell v is occupied
- If a cell is *not* occupied, then we say it is *free*: $Free \coloneqq \neg Occ$
- One may assume that $Occ = \neg\neg Occ$ however this need not hold
- This might appear to be: 1) "annoying" and 2) not useful, however it does enable us to capture some subtle nuances

DOUBLE NEGATION

- Assume that we have three agents A, B and C that can occupy cells in our grid world
- For each agent we can then consider Occ_A, Occ_B and Occ_C namely the temporal landscape that for $v: V$ describes the occupancy of such a cell by a respective agent

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- The English sentence has a slightly ambiguity that is easily distinguished by Occ and $\neg\neg Occ$
- Let then Occ be the disjunction of the three predicates Occ_A, Occ_B and Occ_C then:
  - $Occ(v)$ specifies the time intervals over which a single agent—whether A, B or C—remain in the cell v throughout
  - $\neg\neg Occ(v)$ specifies the time intervals over which there is always at least one agent v, but agents can come and go

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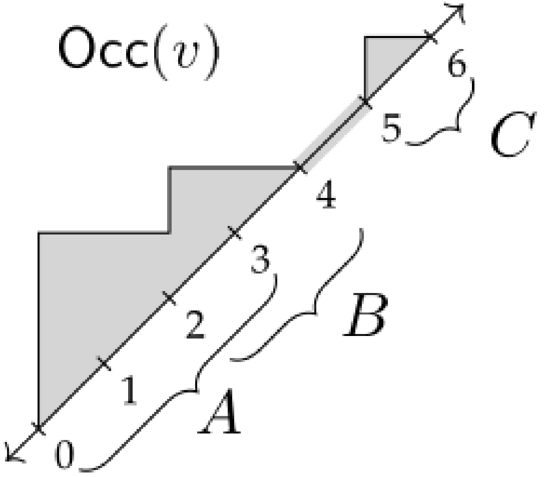
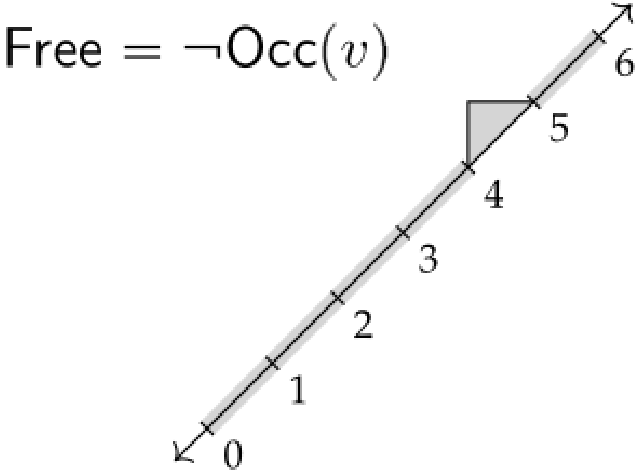
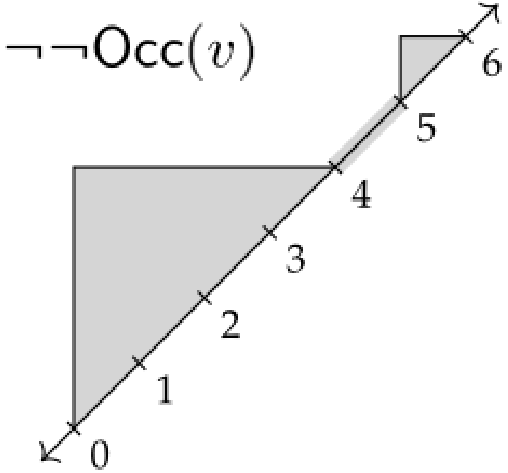
DOUBLE NEGATION: EXAMPLE

- Fix a cell v and let us assume A is in v throughout the interval $[0,3]$, B is in v throughout $[2,4]$ and another agent C is in v throughout [5,6]







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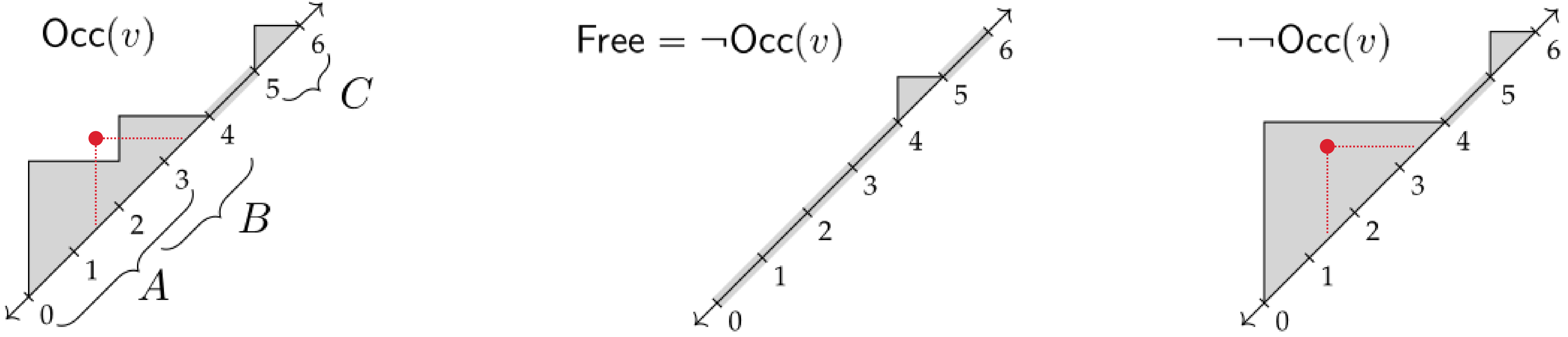

- Note that the temporal landscapes of Occ and $\neg\neg Occ$ are not the same
- Note that Occ does not contain, for example, the interval **[1.5, 3.5]** expressing the refined idea that over such an interval there is not one specific agent in the cell

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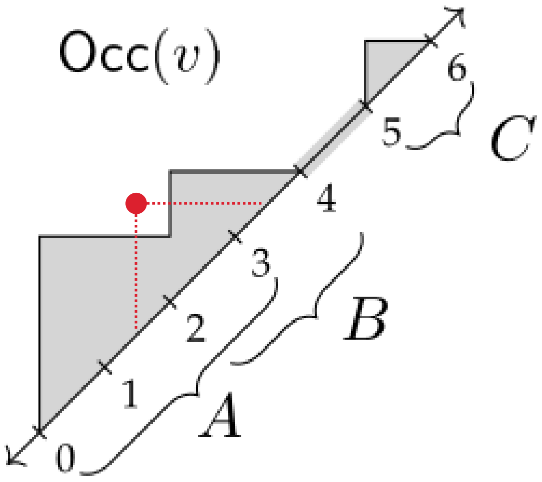

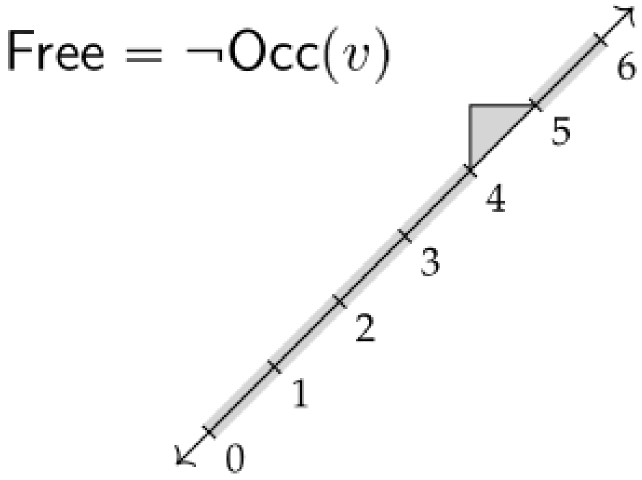

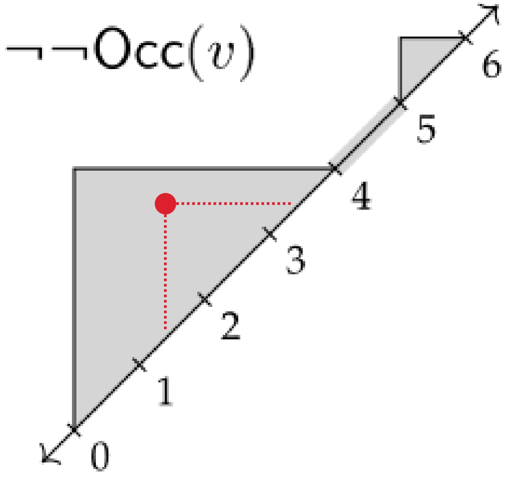

- Then a question one might ask is, why would this be useful in practice?

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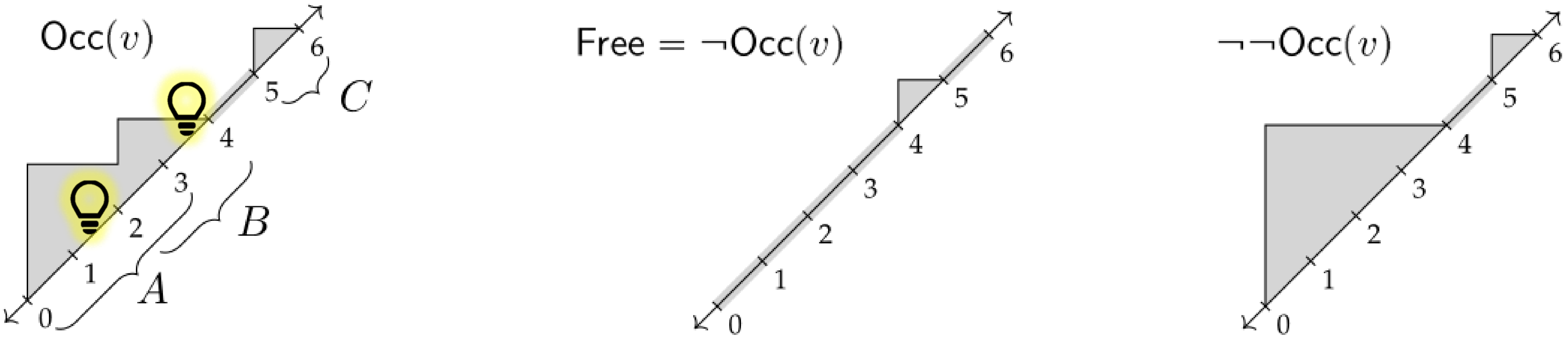
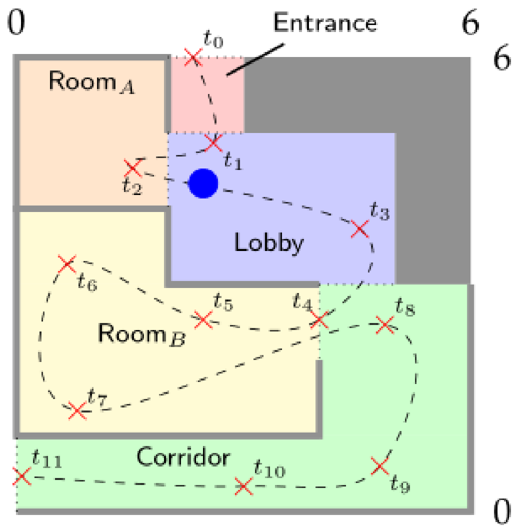


- Then a question one might ask is, why would this be useful in practice?
- Assume that there is a blinking light in the cell v and that two consecutive blinks corresponds to imminent danger
- Then, we see that Occ captures the fact that if the light is ON at time 1.5 and then again at 3.5 the **alarm would be completely missed** unless there is a way for agent A to communicate to agent B the fact that the light was indeed ON at 1.5

CONTINUOUS WORLDS AND QUANTIFIERS

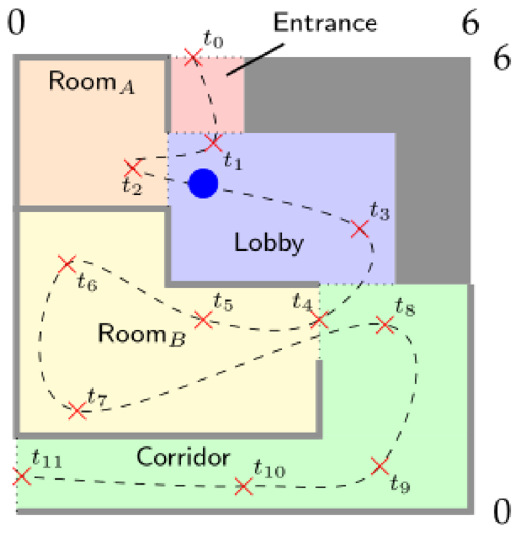


- As we can model discrete (spatial) problems so we can model continuous one
- Now one can define the set of all possible time-parametrized trajectories in the square domain

$$\mathcal{X} \coloneqq \{(x_1, x_2) \in \widetilde{\mathbb{R}} \times \widetilde{\mathbb{R}} \mid 0 \leq x_i \leq 6, i = \{1,2\}\}$$

$\widetilde{\mathbb{R}}$: the behavior type of real numbers continuously changing over any interval (a, b)

CONTINUOUS WORLDS AND QUANTIFIERS



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- We can consider the agent to: 1) have a certain footprint and 2) have a maximum velocity:

$$\text{AgentPos} \coloneqq \left\{ p: \mathcal{X} \to \text{Prop} \;\middle|\; \begin{array}{l} \forall (x_1, x_2 : \mathcal{X}).\left((p(x_1) \wedge p(x_2)) \Rightarrow \text{close}(x_1, x_2)\right) \wedge \\ \forall (x : \mathcal{X}).\, p(x) \Rightarrow (\text{Free}(x) \wedge -v_{\max} \leq \dot{x} \leq v_{\max}) \end{array} \right\}$$

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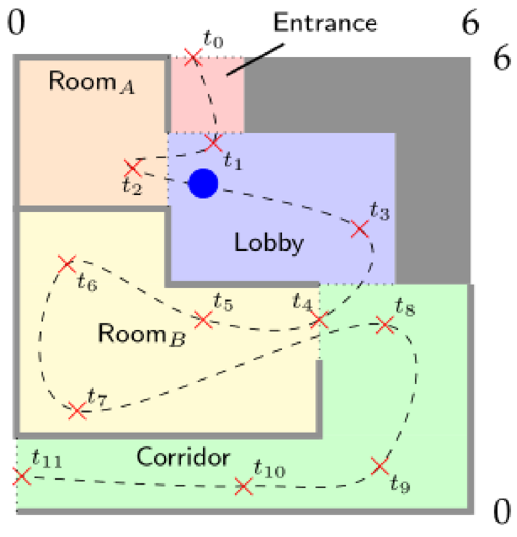

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- Given a constant type $R \coloneqq \{\text{Room}_A, \text{Room}_B, \text{Entrance}, \ldots\}$
- We can then consider the following predicate:

$$\text{AgentInARoom} \coloneqq \exists (r \colon R)\, \forall (x \colon \mathcal{X})\, (\text{Pos}(x) \colon \text{AgentPos}).\, \text{Pos}(x) \Rightarrow \text{Room}(r)(x)$$

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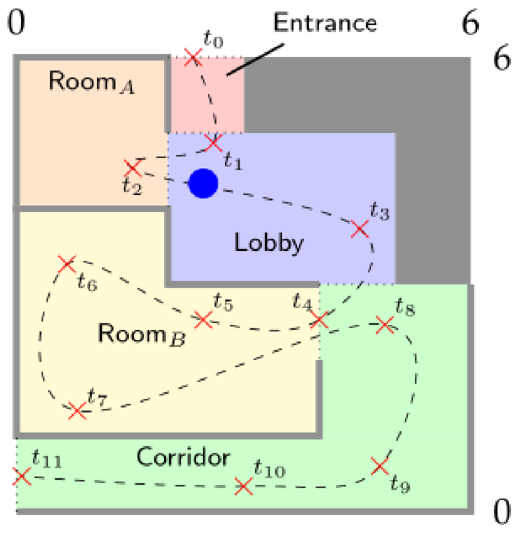
CONTINUOUS WORLDS AND QUANTIFIERS



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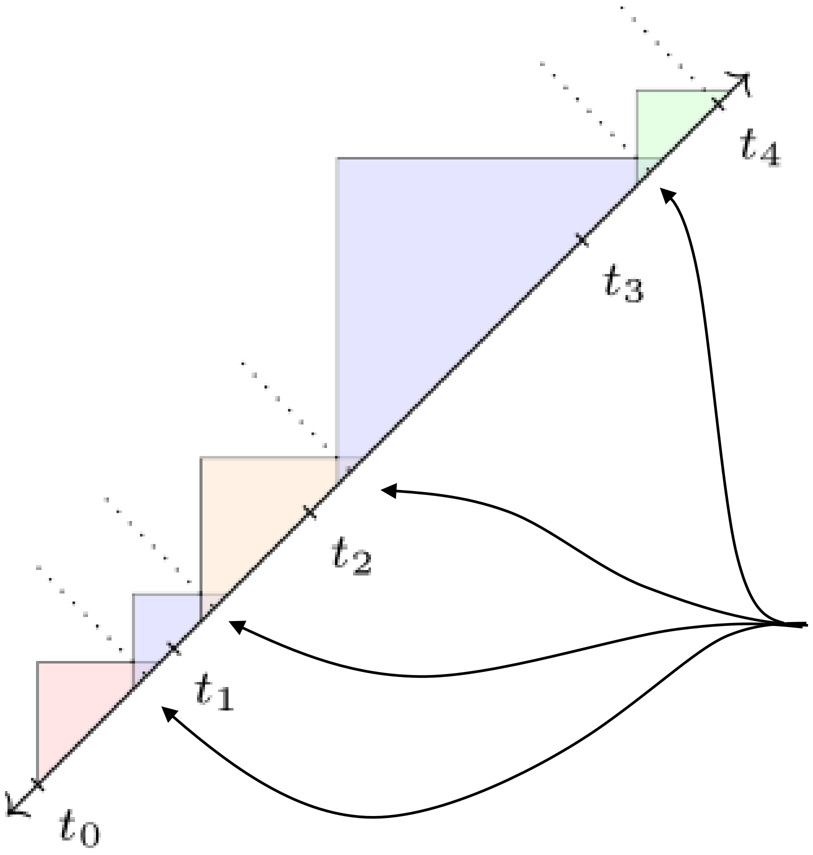
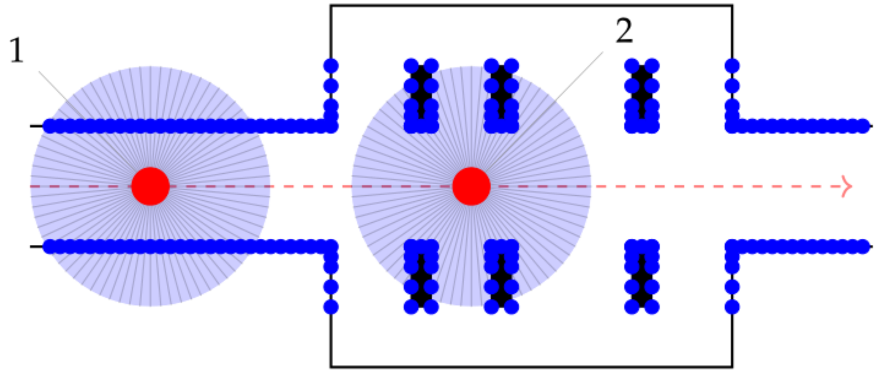


Overlap between rooms caused by the non-zero footprint of the agent

SLANTED TEMPORAL LANDSCAPES



- So far, all the temporal landscapes we have considered have “straight” edges
- Any 1-Lipschitz function defines a temporal landscapes and so one may wonder what practical application could lead to a slanted landscape
- Consider an agent equipped with a spinning LIDAR. Assume a limited storage capacity onboard and the need to be able to reconstruct a map of the environment (store samples)
- The agent is moving at a constant speed in a non-uniform environment, and we consider the predicate:

$$\mathrm{SamplesInMem} = \bigvee_i \mathrm{SampleInMem}(i)$$

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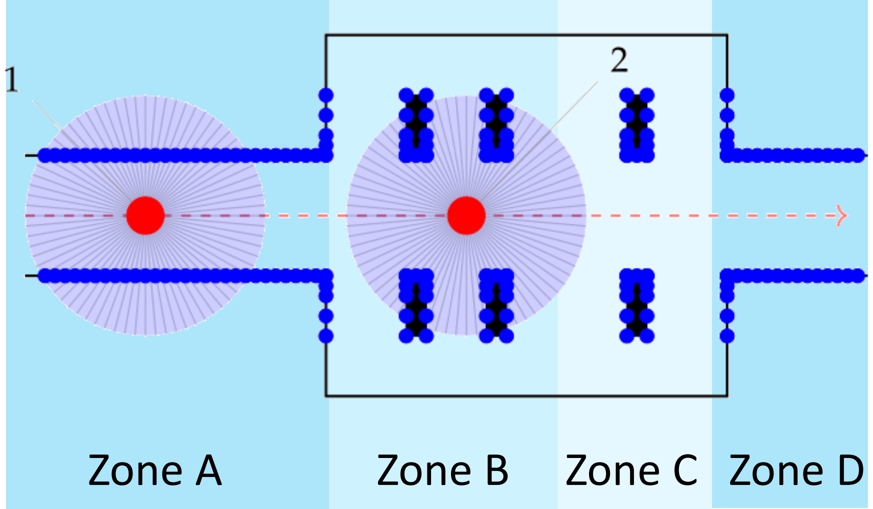
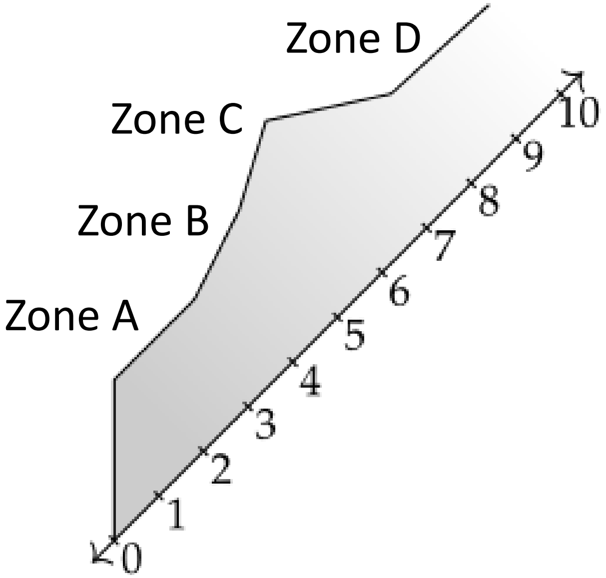


- Then the temporal landscape for a map as in the figure might look like as follows



- In the region of high density of returns samples will be overwritten and will only persist for a maximum amount of time (constant speed of agent and rate of measurements)

- In the region with low density of returns samples will not be overwritten as quickly and as the number of returns samples will persist over longer and longer intervals

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CONCLUSIONS

- Temporal Type Theory helps to reason about complex behaviors over time, however it can be rather difficult to interpret

- Temporal Landscapes provide an intuitive way to visualize predicates describing complex behaviors and reason about their properties

- Temporal Landscapes can be used both when space and/or time are discrete or continuous

- Examples help to clarify the benefit such a visual aid can provide and exemplify the advantage of using an intuitionistic logic