Translating Extensive Form Games to Open Games with Agency

arXiv 2105.06763

Matteo Capucci, Neil Ghani,
Jérémy Ledent, Fredrik Nordvall Forsberg

Mathematically Structured Programming group,
Department of Computer and Information Sciences,
University of Strathclyde

4th International Conference on Applied Category Theory,
Cambridge (UK)

July 14th, 2021

Introduction

Game theory is the mathematical study of interaction among independent, self-interested agents.

– Essentials of Game Theory, Leyton-Brown and Shoham 2008

e.g. economy and ecology, learning, control, etc. $\rightsquigarrow$ *cybernetics.*

Introduction

Game theory is the mathematical study of interaction among independent, self-interested agents.

– Essentials of Game Theory, Leyton-Brown and Shoham 2008

e.g. economy and ecology, learning, control, etc. $\rightsquigarrow$ *cybernetics.*

> **“Does all this have any practical applications?”** It’s really intriguing because this question is asked in almost exactly the same words wherever I go. [...]
>
> I ask them, what do you think constitutes a practical application? [...] Roughly speaking, people converge within five to 10 minutes onto two categories of practical applications. **One is, if you manage to make several million dollars instantly. The other is, if you manage to kill millions of people instantly**. Many people are actually kind of shocked by their own answers.
>
> – Tadashi Tokieda, from Quanta Magazine

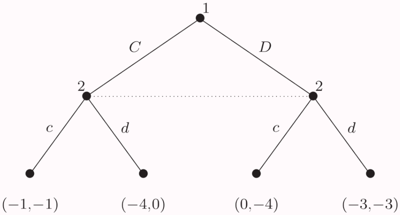
Game theory shows cooperation is hard...

...hopefully better math facilities will make it easier!

Introduction

Classical game theory: **normal form** and **extensive form**

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

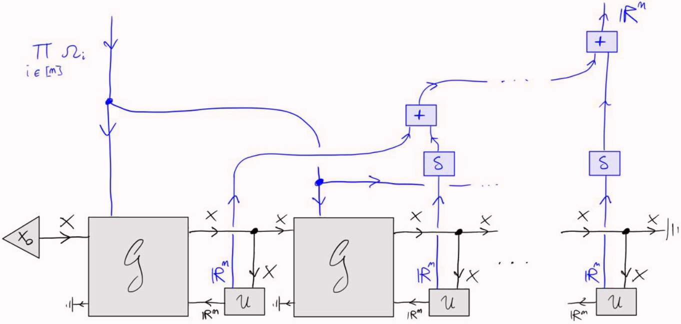


Drawbacks:

1. Too little information (NF)... *how is the game actually played?*
2. Too much information (EF)... *exponential in the number of moves!*
3. Unclear causal relationships (both).
4. Most important
ly: non-compositional! (⇝ small scale)

Introduction

Open games are a categorical framework for compositional game theory[1]:



Advantages:

1. Detailed but not unwieldly... compact notation and true causality
2. Compositional & diagrammatic ($\rightsquigarrow$ large scale).

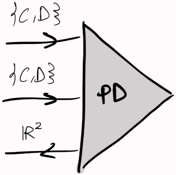
[1] Ghani, Hedges, Winschel, and Zahn 2018

Introduction

Translating normal form games to open games is ‘trivial’...

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

$\rightsquigarrow$



Introduction

Translating extensive form games to open games is non-trivial:

1. We need to seriously consider the problem of agency:
    1. How do we represent **correlated interests** across a game?
    2. How do we represent **player-dependent observational constraints**?
2. We need **adequate composition operators** to reflect the structure of the game.

[2] Capucci, Gavranović, Hedges, and Rischel 2021

Introduction

Translating extensive form games to open games is non-trivial:

1. We need to seriously consider the problem of agency:
    1. How do we represent **correlated interests** across a game?
    2. How do we represent **player-dependent observational constraints**?
2. We need **adequate composition operators** to reflect the structure of the game.

To do so, we introduce:

1. **Open games with agency**, an improved compositional (and conceptual) framework for games, drawing from/inspiring 'open cybernetic systems'[2].
2. An operator calculus for games, in particular new **choice operators**.

[2] Capucci, Gavranović, Hedges, and Rischel 2021

Open games with agency

What is a game?

Game theory is the mathematical study of interaction among independent, self-interested agents.

– Essentials of Game Theory, Leyton-Brown and Shoham 2008

What is a game?

Game theory is the mathematical study of interaction
among independent, self-interested agents.
– Essentials of Game Theory, Leyton-Brown and Shoham 2008

A game factors in two parts:

1. An **arena**, which models the **dynamics** of the game.
2. Some **players**, which intervene in the arena by making **decisions**.

What is a game?

Game theory is the mathematical study of interaction
among independent, self-interested agents.
– Essentials of Game Theory, Leyton-Brown and Shoham 2008

A game factors in two parts:

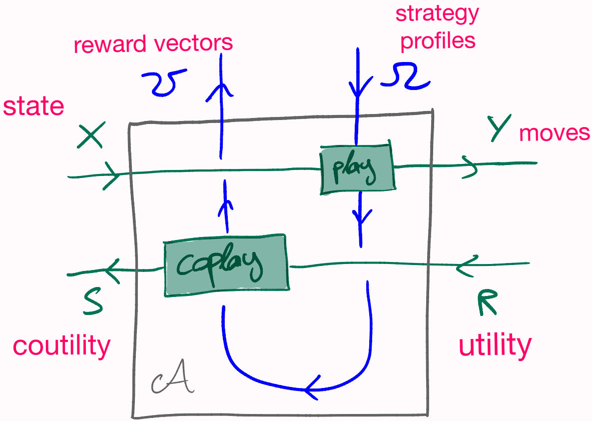
1. An **arena**, which models the **dynamics** of the game.
2. Some **players**, which intervene in the arena by making **decisions**.

A strategy $\omega \in \Omega_p$ for a player $1 \leq p \leq N$ is a policy p uses to make their decisions (e.g. a choice of move for each of p's rounds).

A strategy profile is a strategy for each player $\Omega = \Omega_1 \times \cdots \times \Omega_N$.

What is an arena?

An arena is an open system with three boundaries:



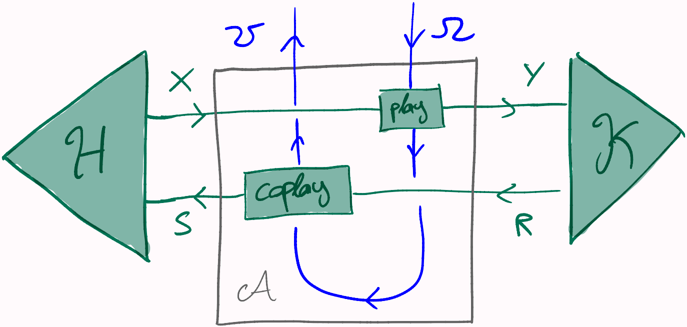
This is a **parametrised lens** $\mathcal{A} : (X, S) \xrightarrow{(\Omega, \mho)} (Y, R)$ specified by two maps

$$\text{play} : \Omega \times X \to Y,$$

$$\text{coplay} : \Omega \times X \times R \to \mho \times S$$

What is an arena?

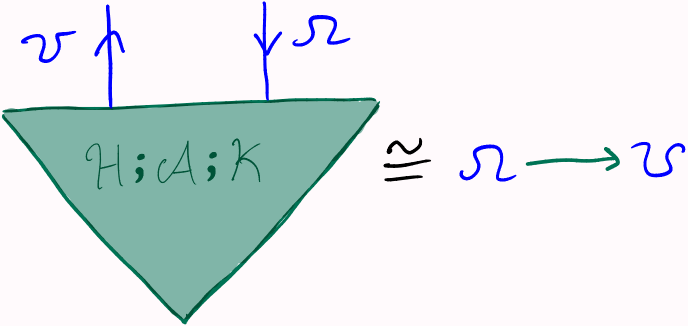
It can be 'closed' by specifying an **initial state** and a **utility function**:



Observe that $\mathsf{Lens}(1,1)(X,S) \cong X$ and $\mathsf{Lens}(Y,R)(1,1) \cong Y \to R$.

What is an arena?

A **closed arena** amounts to an evaluation of strategies with rewards:



What are players?

Players are a *distinct part* of the game (seen as a **system**) which expresses preferences by means of a **selection function**:

$$\varepsilon : (\Omega \to \mho) \longrightarrow \mathsf{P}\Omega$$

In a single-player game, typically Ω is 'finite', $\mho = \mathbb{R}$ and ε is argmax:

$$\mathrm{argmax}(u : \Omega \to \mathbb{R}) = \{\omega \in \Omega \mid \omega \text{ maximises } u\} \subseteq \Omega.$$

What is a Nash equilibrium?

A strategy profile is a **Nash equilibrium**
if no player has incentive to *unilaterally* change strategy.

Traditionally 'the' goal of game theory is determining Nash equilibria of games, though this is not necessarily the case anymore (see: Fudenberg and Levine 1998).

What is a Nash equilibrium?

The assignment $\mathbb{S}(\Omega, \mho) := (\Omega \to \mho) \longrightarrow \mathrm{P}\Omega$ is functorial on Lens.

Crucially, $\mathbb{S}$ admits a lax monoidal structure we call **Nash product**:

$$- \boxtimes - : \mathbb{S}(\Omega_1, \mho_1) \times \mathbb{S}(\Omega_2, \mho_2) \longrightarrow \mathbb{S}(\Omega_1 \times \Omega_2, \mho_1 \times \mho_2)$$

$$(\varepsilon \boxtimes \eta)(u) = \{(\omega_1, \omega_2) \mid \omega_1 \in \varepsilon(u_1(-, \omega_2)) \text{ and } \omega_2 \in \eta(u_2(\omega_1, -))\}$$

where $u = (u_1, u_2) : \Omega_1 \times \Omega_2 \to \mho_1 \times \mho_2$.

What is a Nash equilibrium?

The assignment $\mathbb{S}(\Omega, \mho) := (\Omega \to \mho) \longrightarrow \mathrm{P}\Omega$ is functorial on Lens.

Crucially, $\mathbb{S}$ admits a lax monoidal structure we call **Nash product**:

$$- \boxtimes - : \mathbb{S}(\Omega_1, \mho_1) \times \mathbb{S}(\Omega_2, \mho_2) \longrightarrow \mathbb{S}(\Omega_1 \times \Omega_2, \mho_1 \times \mho_2)$$

$$\begin{aligned}(\varepsilon \boxtimes \eta)(u) &= \{(\omega_1, \omega_2) \mid \omega_1 \in \varepsilon(u_1(-, \omega_2)) \text{ and } \omega_2 \in \eta(u_2(\omega_1, -))\} \\ &= \text{profiles s.t. `no agent wants to unilaterally deviate'}\end{aligned}$$

where $u = (u_1, u_2) : \Omega_1 \times \Omega_2 \to \mho_1 \times \mho_2$.

What is a Nash equilibrium?

The assignment $\mathbb{S}(\Omega, \mho) := (\Omega \to \mho) \longrightarrow \mathrm{P}\Omega$ is functorial on Lens.

Crucially, $\mathbb{S}$ admits a lax monoidal structure we call **Nash product**:

$$- \boxtimes - : \mathbb{S}(\Omega_1, \mho_1) \times \mathbb{S}(\Omega_2, \mho_2) \longrightarrow \mathbb{S}(\Omega_1 \times \Omega_2, \mho_1 \times \mho_2)$$

$$\begin{aligned}(\varepsilon \boxtimes \eta)(u) &= \{(\omega_1, \omega_2) \mid \omega_1 \in \varepsilon(u_1(-, \omega_2)) \text{ and } \omega_2 \in \eta(u_2(\omega_1, -))\} \\ &= \text{profiles s.t. `no agent wants to unilaterally deviate'}\end{aligned}$$

where $u = (u_1, u_2) : \Omega_1 \times \Omega_2 \to \mho_1 \times \mho_2$.

Hence if we have players $P = \{1, \ldots, N\}$, each with preferences $\varepsilon_i \in \mathbb{S}(\Omega_i, \mho_i)$, we get:

$$\varepsilon_1 \boxtimes \cdots \boxtimes \varepsilon_N : (\Omega_1 \times \cdots \times \Omega_N \longrightarrow \mho_1 \times \cdots \times \mho_N) \longrightarrow \mathrm{P}(\Omega_1 \times \cdots \times \Omega_N)$$

What is a Nash equilibrium?

The assignment $\mathbb{S}(\Omega, \mho) := (\Omega \to \mho) \longrightarrow \mathrm{P}\Omega$ is functorial on Lens.

Crucially, $\mathbb{S}$ admits a lax monoidal structure we call **Nash product**:

$$- \boxtimes - : \mathbb{S}(\Omega_1, \mho_1) \times \mathbb{S}(\Omega_2, \mho_2) \longrightarrow \mathbb{S}(\Omega_1 \times \Omega_2, \mho_1 \times \mho_2)$$

$$(\varepsilon \boxtimes \eta)(u) = \{(\omega_1, \omega_2) \,|\, \omega_1 \in \varepsilon(u_1(-, \omega_2)) \text{ and } \omega_2 \in \eta(u_2(\omega_1, -))\}$$

$$= \text{profiles s.t. `no agent wants to unilaterally deviate'}$$

where $u = (u_1, u_2) : \Omega_1 \times \Omega_2 \to \mho_1 \times \mho_2$.

Hence if we have players $P = \{1, \dots, N\}$, each with preferences $\varepsilon_i \in \mathbb{S}(\Omega_i, \mho_i)$, we get:

$$\varepsilon_1 \boxtimes \cdots \boxtimes \varepsilon_N : (\underbrace{\Omega_1 \times \cdots \times \Omega_N}_{\text{strat. profiles}} \longrightarrow \underbrace{\mho_1 \times \cdots \times \mho_N}_{\text{reward vectors}}) \longrightarrow \mathrm{P}(\Omega_1 \times \cdots \times \Omega_N)$$

What is a Nash equilibrium?

The assignment $\mathbb{S}(\Omega, \mho) := (\Omega \to \mho) \longrightarrow \mathrm{P}\Omega$ is functorial on Lens.

Crucially, $\mathbb{S}$ admits a lax monoidal structure we call **Nash product**:

$$- \boxtimes - : \mathbb{S}(\Omega_1, \mho_1) \times \mathbb{S}(\Omega_2, \mho_2) \longrightarrow \mathbb{S}(\Omega_1 \times \Omega_2, \mho_1 \times \mho_2)$$

$$(\varepsilon \boxtimes \eta)(u) = \{(\omega_1, \omega_2) \mid \omega_1 \in \varepsilon(u_1(-, \omega_2)) \text{ and } \omega_2 \in \eta(u_2(\omega_1, -))\}$$

$$= \text{profiles s.t. 'no agent wants to unilaterally deviate'}$$

where $u = (u_1, u_2) : \Omega_1 \times \Omega_2 \to \mho_1 \times \mho_2$.

Hence if we have players $P = \{1, \ldots, N\}$, each with preferences $\varepsilon_i \in \mathbb{S}(\Omega_i, \mho_i)$, we get:

$$\varepsilon_1 \boxtimes \cdots \boxtimes \varepsilon_N : (\overbrace{\underbrace{\Omega_1 \times \cdots \times \Omega_N}_{\text{strat. profiles}} \longrightarrow \underbrace{\mho_1 \times \cdots \times \mho_N}_{\text{reward vectors}}}^{\text{utility functions/}\textbf{closed arenas}}) \longrightarrow \mathrm{P}(\Omega_1 \times \cdots \times \Omega_N)$$

What is a Nash equilibrium?

The assignment $\mathbb{S}(\Omega, \mho) := (\Omega \to \mho) \longrightarrow \mathrm{P}\Omega$ is functorial on Lens.

Crucially, $\mathbb{S}$ admits a lax monoidal structure we call **Nash product**:

$$- \boxtimes - : \mathbb{S}(\Omega_1, \mho_1) \times \mathbb{S}(\Omega_2, \mho_2) \longrightarrow \mathbb{S}(\Omega_1 \times \Omega_2, \mho_1 \times \mho_2)$$

$$\begin{aligned}(\varepsilon \boxtimes \eta)(u) &= \{(\omega_1, \omega_2) \,|\, \omega_1 \in \varepsilon(u_1(-, \omega_2)) \text{ and } \omega_2 \in \eta(u_2(\omega_1, -))\} \\ &= \text{profiles s.t. `no agent wants to unilaterally deviate'}\end{aligned}$$

where $u = (u_1, u_2) : \Omega_1 \times \Omega_2 \to \mho_1 \times \mho_2$.

Hence if we have players $P = \{1, \ldots, N\}$, each with preferences $\varepsilon_i \in \mathbb{S}(\Omega_i, \mho_i)$, we get:

$$\varepsilon_1 \boxtimes \cdots \boxtimes \varepsilon_N : \overbrace{(\underbrace{\Omega_1 \times \cdots \times \Omega_N}_{\text{strat. profiles}} \longrightarrow \underbrace{\mho_1 \times \cdots \times \mho_N}_{\text{reward vectors}})}^{\text{utility functions/}\textbf{closed arenas}} \longrightarrow \underbrace{\mathrm{P}(\Omega_1 \times \cdots \times \Omega_N)}_{\text{Nash equilibria}}$$

The definition

Definition

An **open game with agency** is a pair

$$\mathsf{G} \quad = \quad (\ \mathcal{A} : (X, S) \xrightarrow{(\Omega,\mho)} (Y, R), \quad \varepsilon : \mathbb{S}(\Omega, \mho)\ )$$

whose equilibria are given by

$$\mathrm{eq}_{\mathsf{G}}(x, k) = \varepsilon(x \mathbin{;} \mathcal{A} \mathbin{;} k).$$

In this way we recover the equilibrium predicate of open games (Ghani, Hedges, Winschel, and Zahn 2018).

The definition

Definition

An **open game with agency** is a pair

$$\mathsf{G} \quad = \quad (\ \mathcal{A} : (X, S) \xrightarrow{(\Omega, \mho)} (Y, R), \quad \varepsilon : \mathbb{S}(\Omega, \mho)\ )$$

whose equilibria are given by

$$\mathrm{eq}_{\mathsf{G}}(x, k) = \varepsilon(x \mathbin{⨟} \mathcal{A} \mathbin{⨟} k).$$

In this way we recover the equilibrium predicate of open games (Ghani, Hedges, Winschel, and Zahn 2018).

Definition

G **has set of players** P when

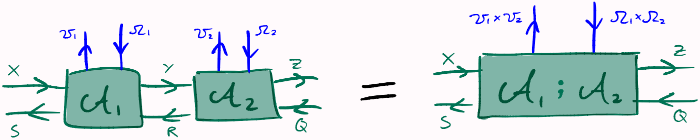
$$\Omega = \prod_{p \in P} \Omega_p, \quad \mho = \prod_{p \in P} \mho_p, \quad \varepsilon = \boxtimes_{p \in P} \varepsilon_p$$

in which case

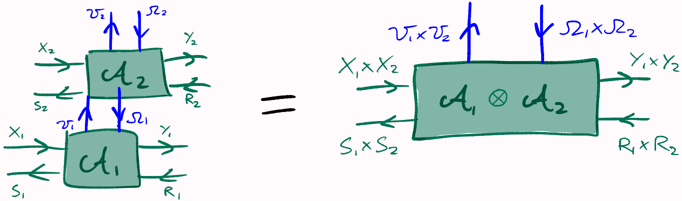
$$\mathrm{eq}_{\mathsf{G}}(x, k) = (\boxtimes_{p \in P} \varepsilon_p)(x \mathbin{⨟} \mathcal{A} \mathbin{⨟} k).$$

Composing games

Sequential composition



Parallel composition

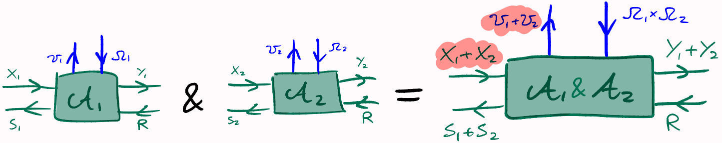


Notice: these operators can be extended to games:

$$(\mathcal{A}_1, \varepsilon) \fatsemi (\mathcal{A}_2, \eta) := (\mathcal{A}_1 \fatsemi \mathcal{A}_2, \varepsilon \boxtimes \eta), \quad (\mathcal{A}_1, \varepsilon) \otimes (\mathcal{A}_2, \eta) := (\mathcal{A}_1 \otimes \mathcal{A}_2, \varepsilon \boxtimes \eta)$$

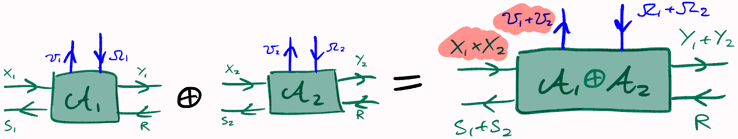
Composing arenas

External choice



The ‘environment’ chooses which game to play, agents are prepared to play both.

Internal choice



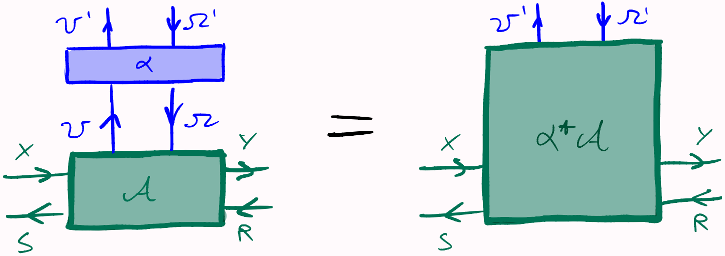
The ‘environment’ can play either game, agents choose which one.

Notice: these operators <u>can't</u> be extended to selection functions in a canonical way!

(Actually $\oplus$ <u>can</u> if we refine our typing judgments)

Reparametrisation

Most importantly arenas form a (locally fibred) **bicategory**: one can **reparametrise** along a lens $\alpha : (\Omega', \mho') \to (\Omega, \mho)$.



This is crucial for introducing agency!

Lenses in the blue direction represent 'players dynamics',
e.g. voting, observational constraints (imperfect information), rewards distribution (imputation), ...

Regrouping

If $\mathcal{A}$ has set of players P and $r : P \to Q$ is a function, we can turn $\mathcal{A}$ into an arena with players Q by reparametrising along the permutation of $\prod_{p \in P} \Omega_P$ induced by r:

$$\mathsf{regroup}_r : \left(\prod_{q \in Q} (r^* \Omega)_q, \prod_{q \in Q} (r^* \mho)_q\right) \longrightarrow \left(\prod_{p \in P} \Omega_p, \prod_{p \in P} \mho_p\right)$$

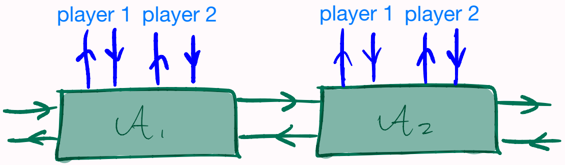
Regrouping

If $\mathcal{A}$ has set of players P and $r : P \to Q$ is a function, we can turn $\mathcal{A}$ into an arena with players Q by reparametrising along the permutation of $\prod_{p \in P} \Omega_P$ induced by r:

$$\mathsf{regroup}_r : (\textstyle\prod_{q\in Q}(r^*\Omega)_q, \prod_{q\in Q}(r^*\mho)_q) \longrightarrow (\prod_{p\in P}\Omega_p, \prod_{p\in P}\mho_p)$$

Example

If $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same players $P = \{1, 2\}$, $\mathcal{A}_1 \fatsemi \mathcal{A}_2$ has players $P + P$.
Regrouping along $\nabla : P + P \to P$ restores the correct set of players.



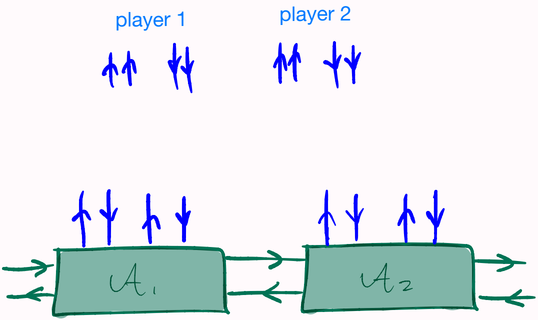
Regrouping

If $\mathcal{A}$ has set of players P and $r : P \to Q$ is a function, we can turn $\mathcal{A}$ into an arena with players Q by reparametrising along the permutation of $\prod_{p \in P} \Omega_P$ induced by r:

$$\mathsf{regroup}_r : (\textstyle\prod_{q \in Q}(r^*\Omega)_q, \prod_{q \in Q}(r^*\mho)_q) \longrightarrow (\prod_{p \in P}\Omega_p, \prod_{p \in P}\mho_p)$$

Example

If $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same players $P = \{1, 2\}$, $\mathcal{A}_1 \fatsemi \mathcal{A}_2$ has players $P + P$. Regrouping along $\nabla : P + P \to P$ restores the correct set of players.



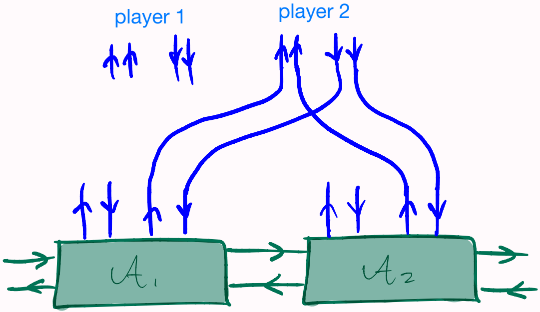
Regrouping

If $\mathcal{A}$ has set of players P and $r : P \to Q$ is a function, we can turn $\mathcal{A}$ into an arena with players Q by reparametrising along the permutation of $\prod_{p \in P} \Omega_P$ induced by r:

$$\mathrm{regroup}_r : (\textstyle\prod_{q \in Q} (r^* \Omega)_q, \prod_{q \in Q} (r^* \mho)_q) \longrightarrow (\prod_{p \in P} \Omega_p, \prod_{p \in P} \mho_p)$$

Example

If $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same players $P = \{1, 2\}$, $\mathcal{A}_1 \fatsemi \mathcal{A}_2$ has players $P + P$. Regrouping along $\nabla : P + P \to P$ restores the correct set of players.



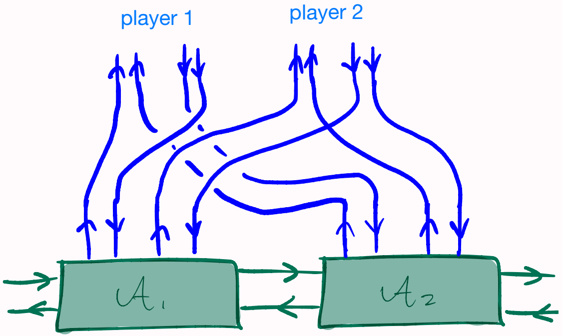
Regrouping

If $\mathcal{A}$ has set of players P and $r : P \to Q$ is a function, we can turn $\mathcal{A}$ into an arena with players Q by reparametrising along the permutation of $\prod_{p \in P} \Omega_P$ induced by r:

$$\mathsf{regroup}_r : (\textstyle\prod_{q \in Q} (r^* \Omega)_q, \prod_{q \in Q} (r^* \mho)_q) \longrightarrow (\prod_{p \in P} \Omega_p, \prod_{p \in P} \mho_p)$$

Example

If $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same players $P = \{1, 2\}$, $\mathcal{A}_1 \fatsemi \mathcal{A}_2$ has players $P + P$. Regrouping along $\nabla : P + P \to P$ restores the correct set of players.



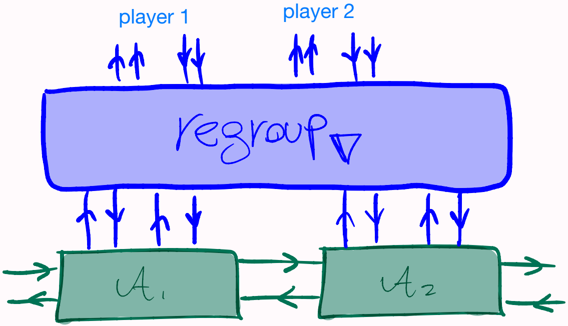
Regrouping

If $\mathcal{A}$ has set of players P and $r : P \to Q$ is a function, we can turn $\mathcal{A}$ into an arena with players Q by reparametrising along the permutation of $\prod_{p \in P} \Omega_P$ induced by r:

$$\mathsf{regroup}_r : (\textstyle\prod_{q \in Q} (r^* \Omega)_q, \prod_{q \in Q} (r^* \mho)_q) \longrightarrow (\prod_{p \in P} \Omega_p, \prod_{p \in P} \mho_p)$$

Example

If $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same players $P = \{1, 2\}$, $\mathcal{A}_1 \fatsemi \mathcal{A}_2$ has players $P + P$. Regrouping along $\nabla : P + P \to P$ restores the correct set of players.

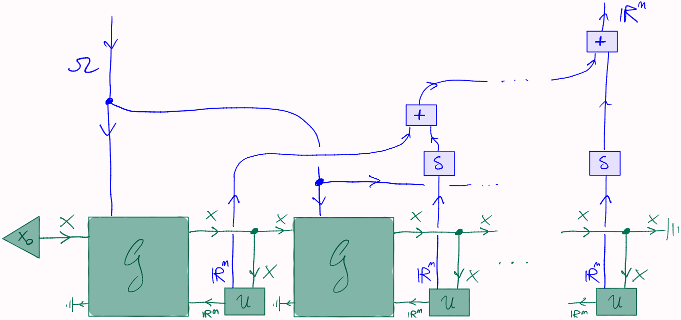


Tying

By reparametrising along $(\Delta, \mathsf{combine})$ (where $\mathsf{combine} = +, \pi_2, \max, \ldots$), we can enforce the same strategies to be played at different points of a game.

Example

Repeated games: play the same strategies at every round

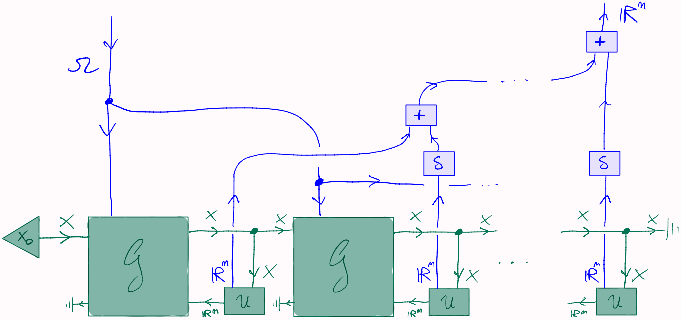


Tying

By reparametrising along $(\Delta, \mathrm{combine})$ (where $\mathrm{combine} = +, \pi_2, \max, \ldots$), we can enforce the same strategies to be played at different points of a game.

Example

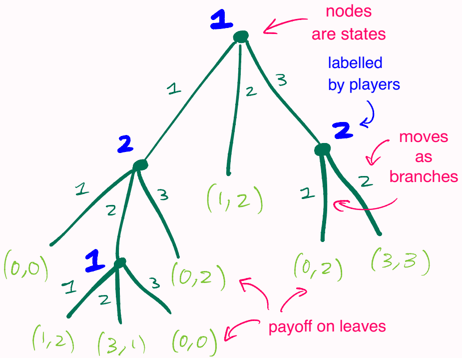
Repeated games: play the same strategies at every round



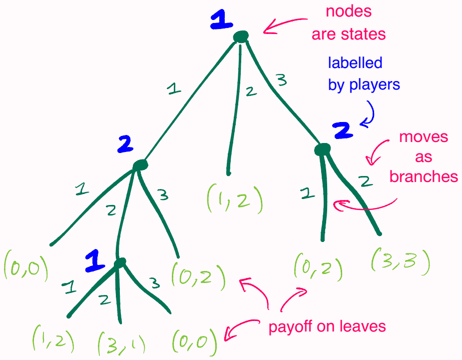
Notice: $(\Delta, \mathrm{combine})^*(\mathcal{A}_1 \fatsemi \mathcal{A}_2)$ lies outside the image of $\fatsemi / \otimes / \oplus / \&$, hence introduces 'non-compositional' effects $\rightsquigarrow$ **'agency is non-local'.**

Extensive form games
& their translation

Extensive form



Extensive form



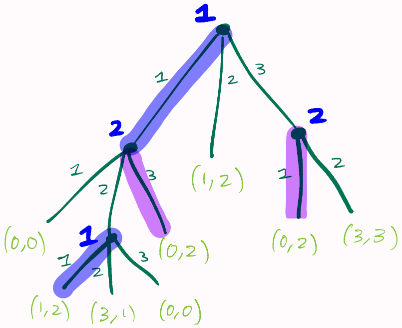
Definition

A **(perfect information) extensive-form tree** is a term of

$$\text{data PETree} = \text{Leaf} \;\; R^P \quad | \quad \text{Node} \;\; P \;\; (n : \mathbb{N}^+) \;\; ([n] \to \text{PETree})$$

where P = set of **players**, R = type of **rewards** (usually $\mathbb{R}$).

Extensive form: strategies



Definition

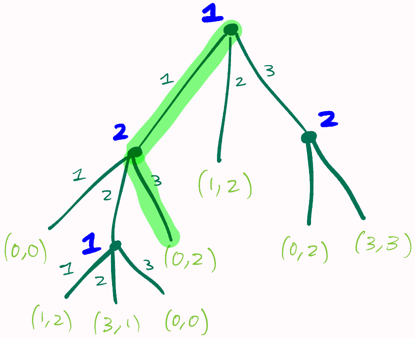
$$\mathsf{strat}_{\mathsf{PET}} : \mathsf{PETree} \to P \to \mathsf{Set}$$

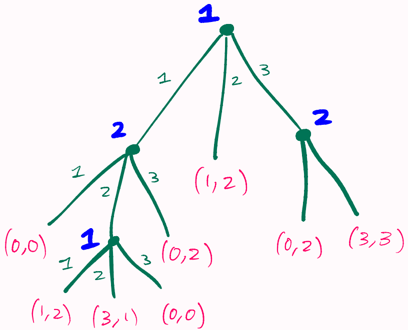
$$\mathsf{strat}_{\mathsf{PET}}\ (\mathsf{Leaf}\ v)\ p = 1$$

$$\mathsf{strat}_{\mathsf{PET}}\ (\mathsf{Node}\ q\ n\ f)\ p$$

$$= (\mathsf{if}\ p \equiv q\ \mathsf{then}\ [n]\ \mathsf{else}\ 1) \times (\Pi\, m \in [n])\ \mathsf{strat}_{\mathsf{PET}}\ (f\ \ m)\ p$$

Extensive form: moves





Definition

$\text{path}_{\text{PET}} : \text{PETree} \to \text{Set}$

$\text{path}_{\text{PET}}\ (\text{Leaf}\ v) = 1$

$\text{path}_{\text{PET}}\ (\text{Node}\ p\ n\ f) =$

$\quad (\Sigma\, m \in [n])\ \text{path}_{\text{PET}}\ (f\ \ m)$

Definition

$\text{payoff}_{\text{PET}} : (T : \text{PETree}) \to (\text{path}_{\text{PET}}\ \ T) \to R^{P}$

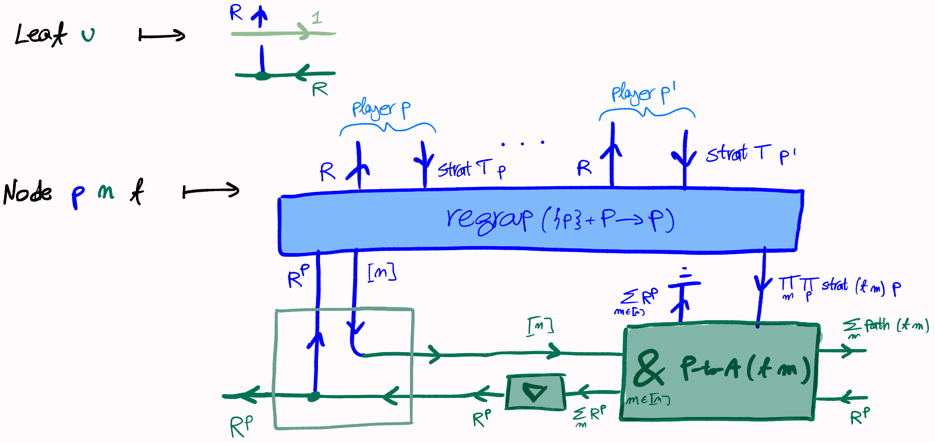
$\text{payoff}_{\text{PET}}\ (\text{Leaf}\ v)\ \bullet = v$

$\text{payoff}_{\text{PET}}\ (\text{Node}\ p\ n\ f)\ (m, \pi)$

$\quad = \text{payoff}_{\text{PET}}\ (f\ \ m)\ \pi$

Translation

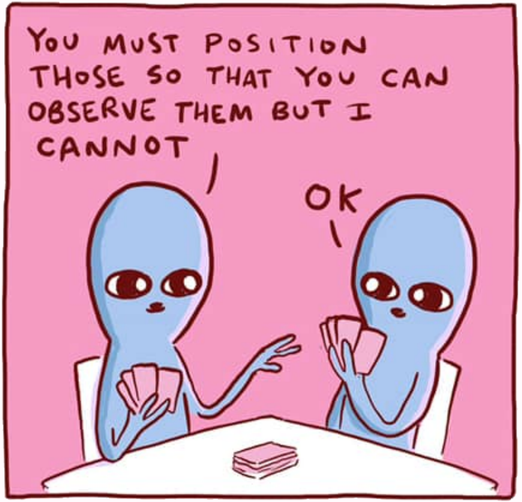
Finally, we can translate any PETree to an open game with agency:



and we equip this arena with $\boxtimes_{p \in P} \operatorname{argmax}(- \fatsemi \pi_p)$.

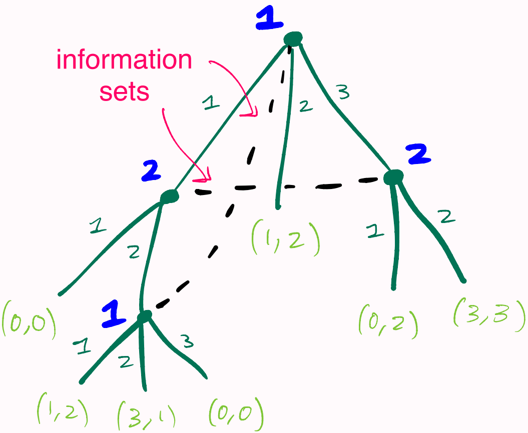
Imperfect information

Sometimes players can't access the whole state of the game (i.e. history of play)



Imperfect information

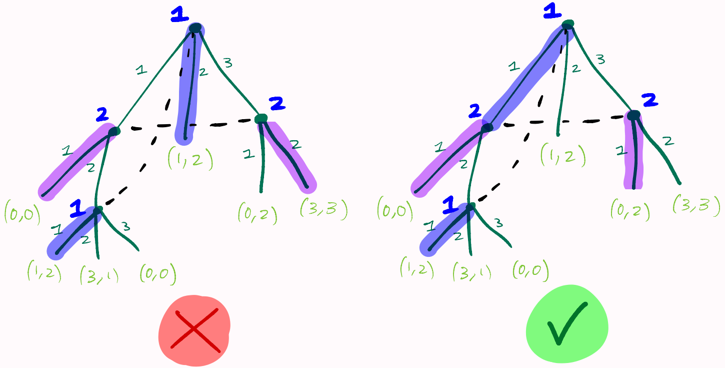
Access to information (or better, lack thereof) is represented by sets of 'indistinguishable states' called **information sets**:



Limit case: all information sets are singletons $\equiv$ perfect information.

Imperfect information

Strategies need to respect information sets:



Players can't distinguish between nodes in the same set, so they play in the same way.

Adding imperfect information

Definition

An **imperfect information extensive-form tree** is a term of

$$\text{data}\ \ \text{IETree} = \text{Leaf}\ \ R^P \mid \text{Node}\ \ (i : I)\ \ ([n\ i] \to \text{IETree})$$

where

1. $I : \text{Set}$ is a set of **information labels**,
2. $n : I \to \mathbb{N}^+$ assigns moves to nodes of the same information set and
3. there is a (surjective) map $\text{belongs} : I \to P$.

Translation

Idea: information sets are instances of 'tying':

1. Forget about information sets and recover a perfect information game:

$$\text{IET-to-PET} : \text{IETree} \to \text{PETree}$$

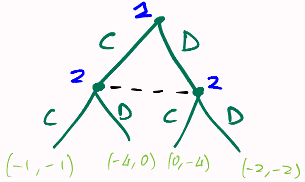
$$\text{IET-to-PET}\ (\text{Leaf}\ v) = \text{Leaf}\ v$$

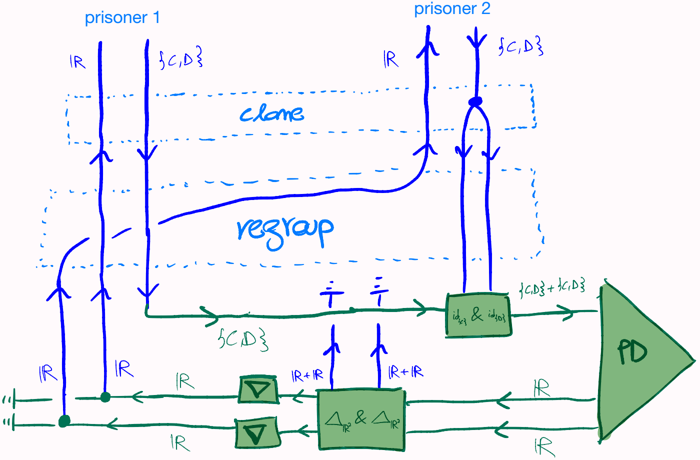
$$\text{IET-to-PET}\ (\text{Node}\ i\ f) = \text{Node}\ (\text{belongs}\ i)\ (n\ i)\ (\lambda m\,.\,\text{IET-to-PET}\ (f\ m))$$

2. Translate it using PET-to-Arena.
3. Reparametrise along clone which ties strategies in the same information sets.

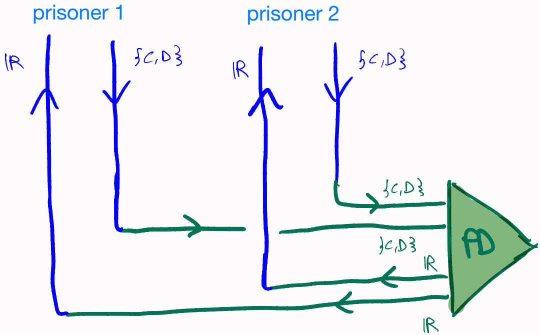
$$\text{IET-to-Arena}(T) = \text{clone}^*(\text{PET-to-Arena}\ (\text{IET-to-PET}\ T))$$

Example: Prisoner's Dilemma





Example: Prisoner's Dilemma



Conclusions

In this work:

1. We have seen how games naturally decompose in *arenas* and *selection functions*, with *reparametrisations* playing a key role
2. This allows to handle **long-range correlations** in players' behaviour
3. Hence we can easily map extensive form trees to open games with agency

Future directions

There remain some open questions related to the EF$\rightarrow$OG translation:

1. Can we simplify the resulting game using topological moves?
   e.g. translating IEF often yields OG which are $\otimes$-decomposable
2. Can we treat subgame-perfect equilibrium?
   Can Escardó-Oliva product of selections[3] be a lax monoidal structure on $\mathbb{S}$?

[3] Escardó and Oliva 2015

Future directions

There remain some open questions related to the EF$\to$OG translation:

1. Can we simplify the resulting game using topological moves?
   e.g. translating IEF often yields OG which are $\otimes$-decomposable
2. Can we treat subgame-perfect equilibrium?
   Can Escardó-Oliva product of selections[3] be a lax monoidal structure on $\mathbb{S}$?

Moreover, the framework of open games (with agency) is still incomplete.

1. Selection functions do not interact much with arenas
   Is there a better way to equip arenas with equilibrium predicates?
2. What kind of assignment is '$\mathrm{argmax} : \mathrm{Arenas} \to \mathrm{Games}$'?
   An oplax Para coalgebra?

[3] Escardó and Oliva 2015

Thanks for your attention!

Questions?

References I

K. Leyton-Brown and Y. Shoham, "Essentials of game theory: A concise multidisciplinary introduction", *Synthesis lectures on artificial intelligence and machine learning*, vol. 2, no. 1, pp. 1–88, 2008.

N. Ghani, J. Hedges, V. Winschel, and P. Zahn, "Compositional game theory", in *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, 2018, pp. 472–481.

M. Capucci, B. Gavranović, J. Hedges, and E. F. Rischel, "Towards foundations of categorical cybernetics", *arXiv preprint arXiv:2105.06332*, 2021.

D. Fudenberg and D. K. Levine, *The theory of learning in games*. MIT press, 1998, vol. 2.

M. Escardó and P. Oliva, "Bar recursion and products of selection functions", *The Journal of Symbolic Logic*, vol. 80, no. 1, pp. 1–28, 2015.