Frey, Zwanziger — $(1,2)$-cosmoses and tagged linear logic

Recall: An **(elementary) topos** is a category $\mathcal{E}$ with finite limits such that all presheaves $\mathbf{Rel}_{\mathcal{E}}(-, A) : \mathcal{E}^{\mathrm{op}} \to \mathbf{Set}$ are representable.

Definition

A $(1,2)$**-cosmos** is a **Pos**-enriched category $\mathcal{E}$ with finite limits, such that all

$$\mathbf{Prof}_{\mathcal{E}}(-, A) : \mathcal{E}^{\mathrm{coop}} \to \mathbf{Pos} \quad \text{and} \quad \mathbf{Prof}_{\mathcal{E}}(A, -) : \mathcal{E}^{\mathrm{op}} \to \mathbf{Pos}$$

are representable.

- representing objects are *lower* and *upper power objects* $P_{\downarrow}A$, $P_{\uparrow}A$
- monadicity fails – $(P_{\downarrow} \circ P_{\uparrow})$-algebras on **Pos** are *completely dist. lattices*

virtual double category $\mathbf{Prof}(\mathcal{E})$ — *tagged linear logic with substitutions*

$$\begin{array}{ccccccc} B_0 & & \xleftarrow{\quad\quad\psi\quad\quad} & & B_1 & & \\ f\uparrow & & & & \uparrow g & \quad — \quad & \phi_1, \ldots, \phi_n \vdash \psi[f, g] \\ A_0 & \xleftarrow{\phi_1} & A_1 \quad \cdots \quad A_{n-1} & \xleftarrow{\phi_n} & A_n & & \end{array}$$

Thm: $\mathbf{Prof}(\mathcal{E})$ is *closed* and has *compositions*; $\mathbf{Prof}_{\mathcal{E}}(A, 1)$ is Heyting alg.