Polymorphic Automorphisms and the Picard Group

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Introduction

- The notion of *definable automorphism* occurs throughout algebra, model theory, and computer science.
- In first-order logic, an automorphism α of a model M of a first-order theory is *definable* if there is a formula $\varphi(x, y)$ such that $\alpha(a) = b$ iff $M \models \varphi(a, b)$ for all $a, b \in M$. E.g. if G is a group, then the inner automorphism induced by $g \in G$ is definable by the formula $y = gxg^{-1}$.
- Definable automorphisms are *polymorphic* or *uniform*, and can provide a generalized notion of *inner automorphism*.

Motivation

- To motivate this, we recall that George Bergman proved in [1] that the definable group automorphisms, i.e. the inner automorphisms given by conjugation, can be characterized purely *categorically* as the automorphisms that extend naturally along any group homomorphism.

- To see this, observe first that if α is an inner automorphism of a group G (induced by $s \in G$), then for each group morphism $f : G \to H$ with domain G we can 'push forward' α to define an inner automorphism

$$\alpha_f : H \xrightarrow{\sim} H$$

by conjugation with $f(s) \in H$ (so that $\alpha_{\mathbf{id}_G} = \alpha$).

Motivation

- This family of automorphisms $(\alpha_f)_f$ is *coherent*, in the sense that it satisfies the following *naturality* property: if $f : G \to G'$ and $f' : G' \to G''$ are group homomorphisms, then the following diagram commutes:

$$
\begin{array}{ccc}
G' & \xrightarrow{\alpha_f} & G' \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f'} \\
G'' & \xrightarrow[\alpha_{f' \circ f}]{} & G''
\end{array}
$$

Bergman's Theorem

For a group G, let us call an *arbitrary* family of automorphisms

$$\left(\alpha_f : \mathbf{cod}(f) \xrightarrow{\sim} \mathbf{cod}(f)\right)_{\mathbf{dom}(f)=G}$$

with the above naturality property an *extended inner automorphism* of G. Such a family is a natural automorphism of $G/\mathbf{Group} \to \mathbf{Group}$.

Theorem (Bergman [1])

Let G be a group and $\alpha : G \xrightarrow{\sim} G$ an automorphism of G. Then α is an **inner** *automorphism of G iff there is an extended inner automorphism $(\alpha_f)_f$ of G with $\alpha = \alpha_{\mathbf{id}_G}$.*

This provides a completely *categorical* characterization of inner automorphisms of groups: they are exactly those group automorphisms that are 'coherently extendible' along morphisms out of their domain.

Covariant Isotropy

- We have a functor $\mathcal{Z} : \mathbf{Group} \to \mathbf{Group}$ that sends any group G to its group of extended inner automorphisms $\mathcal{Z}(G)$. We refer to $\mathcal{Z}$ as the *covariant isotropy group (functor)* of the category **Group**. (Bergman's theorem actually entails that $\mathcal{Z} \cong \mathsf{Id} : \mathbf{Group} \to \mathbf{Group}$.)

- In fact, *any* category $\mathbb{C}$ has a *covariant isotropy group (functor)*

$$\mathcal{Z}_{\mathbb{C}} : \mathbb{C} \to \mathbf{Group}$$

that sends each object $C \in \mathbb{C}$ to the group of extended inner automorphisms of C, i.e. families of automorphisms

$$\left(\alpha_f : \mathbf{cod}(f) \xrightarrow{\sim} \mathbf{cod}(f)\right)_{\mathbf{dom}(f)=C}$$

in $\mathbb{C}$ with the same naturality property as before, i.e. natural automorphisms of the projection functor $C/\mathbb{C} \to \mathbb{C}$.

Covariant Isotropy

- We can also turn Bergman's characterization of inner automorphisms in **Group** into a *definition* of inner automorphisms in an arbitrary category $\mathbb{C}$: if $C \in \mathbb{C}$, we say that an automorphism $\alpha : C \xrightarrow{\sim} C$ is *inner* if there is an extended inner automorphism $(\alpha_f)_f \in \mathcal{Z}_{\mathbb{C}}(C)$ with $\alpha_{\mathbf{id}_C} = \alpha$.

- Notice that **Group** is the category of (set-based) *models* of an *algebraic theory*, i.e. a set of equational axioms between terms, namely the theory $\mathbb{T}_{\mathbf{Grp}}$ of groups. So $\mathbf{Group} = \mathbb{T}_{\mathbf{Grp}}\mathbf{mod}$.

- We will generalize ideas from the proof of Bergman's Theorem to give a 'syntactic' characterization of the (extended) inner automorphisms of $\mathbb{T}\mathbf{mod}$, i.e. of the covariant isotropy group of $\mathbb{T}\mathbf{mod}$, for any so-called *quasi-equational* theory $\mathbb{T}$.

Covariant Isotropy

- We will then use this result to characterize the covariant isotropy groups of the category **StrMonCat** of strict monoidal categories and any presheaf category $\mathbf{Set}^{\mathcal{J}}$.
- In particular, we will show that the covariant isotropy group of **StrMonCat** sends any strict monoidal category to its *Picard group*, i.e. its group of $\otimes$-invertible objects.

Quasi-Equational Theories

- What is a quasi-equational theory? (Also known as: partial Horn theory, essentially algebraic theory, cartesian theory, finite limit theory.)
- First, we need the notion of a *signature* Σ, which consists of a non-empty set $\Sigma_{\mathbf{Sort}}$ of *sorts*, and a set $\Sigma_{\mathbf{Fun}}$ of (typed) *function/operation symbols*.
- For example, the signature for *groups* has one sort X and three function symbols $\cdot : X \times X \to X$, ${}^{-1} : X \to X$, and $e : X$. The signature for *categories* has two sorts O, A and four function symbols $\mathbf{dom}, \mathbf{cod} : A \to O$, $\mathbf{id} : O \to A$, and $\circ : A \times A \to A$.

Quasi-Equational Theories

- We can then form the set **Term**(Σ) of *terms* over Σ, constructed from variables and function symbols, as well as the set **Horn**(Σ) of *Horn formulas* over Σ, which are finite conjunctions of equations between terms.
- A *quasi-equational theory* over a signature Σ is then a set of *implications* (the *axioms* of $\mathbb{T}$) of the form $\varphi \Rightarrow \psi$, with $\varphi, \psi \in$ **Horn**(Σ) (see [7]).
- The operation symbols of a quasi-equational theory are only required to be *partially* defined. If t is a term, we write $t\downarrow$ as an abbreviation for $t = t$, meaning 't is defined'.

Examples

- Any *algebraic* theory, whose axioms all have the form $\top \Rightarrow \psi$, where $\top$ is the empty conjunction. E.g. the theories of sets, semigroups, (commutative) monoids, (abelian) groups, (commutative) rings with unit, etc.

- The theories of categories and groupoids. E.g. two of the axioms of the theory of categories are

$$g \circ f \downarrow \Rightarrow \mathbf{dom}(g) = \mathbf{cod}(f),$$

$$\mathbf{dom}(g) = \mathbf{cod}(f) \Rightarrow g \circ f \downarrow .$$

- The theory of strict monoidal categories, and the theory of presheaves $\mathcal{J} \to \mathbf{Set}$ on a small category $\mathcal{J}$.

Proof of Bergman's Theorem

- To motivate our characterization of covariant isotropy for categories of models of quasi-equational theories, let us review a specific idea in the proof of Bergman's Theorem.

- Consider the group $G\langle \mathbf{x} \rangle$ obtained from a group G by freely adjoining an indeterminate element $\mathbf{x}$. Elements of $G\langle \mathbf{x} \rangle$ are (reduced) group words in $\mathbf{x}$ and elements of G.

- The underlying set of $G\langle \mathbf{x} \rangle$ can be endowed with a *substitution monoid* structure: given $w_1, w_2 \in G\langle \mathbf{x} \rangle$, we set $w_1 \cdot w_2$ to be the reduction of $w_1[w_2/\mathbf{x}]$, and the unit is $\mathbf{x}$ itself.

- If $w \in G\langle \mathbf{x} \rangle$, w *commutes generically with* the group operations if:
  - In $G\langle \mathbf{x}_1, \mathbf{x}_2 \rangle$, the reduction of $w[\mathbf{x}_1/\mathbf{x}]w[\mathbf{x}_2/\mathbf{x}]$ is $w[\mathbf{x}_1\mathbf{x}_2/\mathbf{x}]$;
  - In $G\langle \mathbf{x} \rangle$, the reduction of w^{-1} is $w\left[\mathbf{x}^{-1}/\mathbf{x}\right]$;
  - In $G\langle \mathbf{x} \rangle$, the reduction of $w[e/\mathbf{x}]$ in $G\langle \mathbf{x} \rangle$ is e.

Proof of Bergman's Theorem

- E.g. if $g \in G$, then the word $g\mathbf{x}g^{-1} \in G\langle \mathbf{x} \rangle$ commutes generically with the group operations:
  - $g\mathbf{x}_1 g^{-1} g\mathbf{x}_2 g^{-1} \sim g\mathbf{x}_1\mathbf{x}_2 g^{-1}$
  - $\left(g\mathbf{x}g^{-1}\right)^{-1} \sim \left(g^{-1}\right)^{-1}\mathbf{x}^{-1}g^{-1} \sim g\mathbf{x}^{-1}g^{-1}$,
  - $geg^{-1} \sim gg^{-1} \sim e$.
- Let $\mathbf{Inv}(G\langle \mathbf{x} \rangle)$ be the subgroup of *invertible* elements of the substitution monoid $G\langle \mathbf{x} \rangle$. (E.g. $g\mathbf{x}g^{-1}$ is invertible, with inverse $g^{-1}\mathbf{x}g$.)
- Then the proof of Bergman's Theorem shows that the group $\mathcal{Z}(G)$ is isomorphic to the subgroup of $\mathbf{Inv}(G\langle \mathbf{x} \rangle)$ consisting of all words that commute generically with the group operations.

The Isotropy Group of a Quasi-Equational Theory

- Fix a quasi-equational theory $\mathbb{T}$ over a signature Σ, and let $\mathbb{T}\textbf{mod}$ be the category of (set-based) models of $\mathbb{T}$.

- We will now give a *logical/syntactic* characterization of the covariant isotropy group

$$\mathcal{Z}_{\mathbb{T}} : \mathbb{T}\textbf{mod} \to \textbf{Group}$$

  of $\mathbb{T}\textbf{mod}$.

- Using the quasi-equational syntax of $\mathbb{T}$, we can define a notion of *definable automorphism* for a model M of $\mathbb{T}$, and the definable automorphisms of any $M \in \mathbb{T}\textbf{mod}$ form a group $\textbf{DefInn}(M)$.

Definable Automorphisms

- Given $M \in \mathbb{T}\mathbf{mod}$ and $A \in \Sigma_{\mathbf{Sort}}$, one can form the $\mathbb{T}$-model $M\langle \mathbf{x}_A \rangle$ obtained from M by freely adjoining an indeterminate element $\mathbf{x}_A$ of sort A. For any sort B, elements of $M\langle \mathbf{x}_A \rangle_B$ are congruence classes $[t]$ of Σ-terms t of sort B involving $\mathbf{x}_A$ and constants from M, where two such terms s, t are congruent if they are provably equal in the *diagram theory* $\mathbb{T}(M, \mathbf{x}_A)$ of M extended by the axiom $\top \vdash \mathbf{x}_A \downarrow$.
- For any sort A, the set $M\langle \mathbf{x}_A \rangle_A$ is a monoid under substitution: the unit is $[\mathbf{x}_A]$ and $[s] \cdot [t] = [s\,[t/\mathbf{x}_A]]$ for $[s], [t] \in M\langle \mathbf{x}_A \rangle_A$. We then have the product monoid $\prod_A M\langle \mathbf{x}_A \rangle_A$.

Definable Automorphisms

- So an element $([s_A])_A \in \prod_A M\langle \mathbf{x}_A \rangle_A$ is *(substitutionally) invertible* if for each sort A, there is some $\left[s_A^{-1}\right] \in M\langle \mathbf{x}_A \rangle_A$ with

$$\mathbb{T}(M, \mathbf{x}_A) \vdash s_A\left[s_A^{-1}/\mathbf{x}_A\right] = \mathbf{x}_A = s_A^{-1}\left[s_A/\mathbf{x}_A\right].$$

- If $f : A_1 \times \ldots \times A_n \to A$ is an operation symbol of Σ, then $([s_A])_A \in \prod_A M\langle \mathbf{x}_A \rangle_A$ *commutes generically with* f if $\mathbb{T}\left(M, \mathbf{x}_{A_1}, \ldots, \mathbf{x}_{A_n}\right)$ proves the sequent

$$f\left(\mathbf{x}_{A_1}, \ldots, \mathbf{x}_{A_n}\right)\downarrow \vdash s_A\left[f\left(\mathbf{x}_{A_1}, \ldots, \mathbf{x}_{A_n}\right)/\mathbf{x}_A\right] = f\left(s_{A_1}, \ldots, s_{A_n}\right),$$

and *reflects definedness of* f if $\mathbb{T}\left(M, \mathbf{x}_{A_1}, \ldots, \mathbf{x}_{A_n}\right)$ proves the sequent

$$s_A\left[f\left(\mathbf{x}_{A_1}, \ldots, \mathbf{x}_{A_n}\right)/\mathbf{x}_A\right]\downarrow \vdash f\left(\mathbf{x}_{A_1}, \ldots, \mathbf{x}_{A_n}\right)\downarrow.$$

Definable Automorphisms

We define **DefInn**(M) to be the subgroup of the product monoid $\prod_A M\langle \mathbf{x}_A \rangle_A$ consisting of the invertible elements that commute generically with and reflect definedness of every $f \in \Sigma_{\mathbf{Fun}}$.

Theorem ([4])

Let $\mathbb{T}$ be a quasi-equational theory. For any $M \in \mathbb{T}\mathbf{mod}$, the covariant isotropy group $\mathcal{Z}_{\mathbb{T}}(M)$, i.e. the group of extended inner automorphisms of M, is isomorphic to the group **DefInn**(M) *of definable automorphisms of M.*

In particular, an automorphism $\alpha : M \xrightarrow{\sim} M$ in $\mathbb{T}\mathbf{mod}$ is *inner* iff there is some $([s_A])_A \in \mathbf{DefInn}(M)$ that *induces* α, i.e. for each sort A

$$(m \in M_A) \qquad \alpha_A(m) = s_A\,[m/\mathbf{x}_A]^M \in M_A.$$

Initial Examples ([3])

- If $\mathbb{T}$ is the theory of sets, then $\mathbb{T}$ has trivial isotropy group, i.e. $\mathcal{Z}_{\mathbb{T}}(S) \cong \mathbf{DefInn}(S) \cong \{[\mathbf{x}]\}$ for any set S, so the only inner automorphism of a set is the *identity* function.

- If $\mathbb{T}$ is the theory of groups, then Bergman proved $\forall G \in \mathbb{T}\mathbf{mod} = \mathbf{Group}$ that

$$\mathcal{Z}_{\mathbb{T}}(G) \cong \mathbf{DefInn}(G) \cong \left\{ \left[g\mathbf{x}g^{-1} \right] \in G\langle \mathbf{x} \rangle \ \mid g \in G \right\} \cong G.$$

- If $\mathbb{T}$ is the theory of monoids, then $\forall M \in \mathbb{T}\mathbf{mod} = \mathbf{Mon}$ we have

$$\mathcal{Z}_{\mathbb{T}}(M) \cong \mathbf{DefInn}(M) \cong \left\{ \left[m\mathbf{x}m^{-1} \right] \in M\langle \mathbf{x} \rangle \mid m \in \mathbf{Inv}(M) \right\} \cong \mathbf{Inv}(M).$$

Initial Examples ([3])

- If $\mathbb{T}$ is the theory of abelian groups, then $\forall G \in \mathbb{T}\mathbf{mod} = \mathbf{Ab}$ we have

$$\mathcal{Z}_{\mathbb{T}}(G) \cong \mathbf{DefInn}(G) \cong \{[\mathbf{x}], [-\mathbf{x}]\} \cong \mathbb{Z}_2.$$

- If $\mathbb{T}$ is the theory of commutative monoids or unital rings, then $\mathbb{T}$ has trivial isotropy group.

- If $\mathbb{T}$ is the theory of (not necessarily commutative) unital rings, then $\forall R \in \mathbb{T}\mathbf{mod} = \mathbf{Ring}$ we have

$$\mathcal{Z}_{\mathbb{T}}(R) \cong \mathbf{DefInn}(R) \cong \left\{ \left[r\mathbf{x}r^{-1} \right] \in R\langle \mathbf{x} \rangle \mid r \in \mathbf{Unit}(R) \right\} \cong \mathbf{Unit}(R).$$

- If $\mathbb{T}$ is the theory of categories or groupoids, then $\mathbb{T}$ has trivial isotropy group.

Strict Monoidal Categories

- If $\mathbb{T}$ is the quasi-equational theory of strict monoidal categories, then we proved in [4] that for any strict monoidal category $\mathbb{C}$, the group $\mathbf{DefInn}(\mathbb{C})$ consists (up to isomorphism) of exactly the monoidal *inner* automorphisms, i.e. the automorphisms $F : \mathbb{C} \to \mathbb{C}$ for which there is some $\otimes$-invertible object $c \in \mathbb{C}$ such that F is given by *conjugation* with c, i.e.

$$(a \in \mathbb{C}) \qquad F(a) = c \otimes a \otimes c^{-1}.$$

- We then deduced that

$$\mathcal{Z}_{\mathbb{T}}(\mathbb{C}) \cong \mathbf{DefInn}(\mathbb{C}) \cong \mathbf{Inv}\left(\mathbf{Ob}(\mathbb{C})\right),$$

the group of $\otimes$-invertible elements of the object monoid of $\mathbb{C}$, also known as the *Picard group* of $\mathbb{C}$.

Presheaf Categories

- We can also characterize the covariant isotropy group of a *presheaf category* $\mathbf{Set}^{\mathcal{J}}$ for a small category $\mathcal{J}$.

- Given a small category $\mathcal{J}$, we can define a quasi-equational theory $\mathbb{T}^{\mathcal{J}}$ whose models are functors $\mathcal{J} \to \mathbf{Set}$, i.e.

$$\mathbb{T}^{\mathcal{J}}\mathbf{mod} \cong \mathbf{Set}^{\mathcal{J}}.$$

- The sorts are the objects of $\mathcal{J}$, for any morphism $f : i \to j$ one introduces a unary operation symbol $\widehat{f} : i \to j$, and one has axioms expressing functoriality.

Presheaf Categories

- If $F : \mathcal{J} \to \mathbf{Set}$ is a presheaf, we showed in [4] that $\mathbf{DefInn}(F)$ consists (up to isomorphism) of exactly the natural automorphisms $\alpha : F \xrightarrow{\sim} F$ induced by some element $\psi \in \mathbf{Aut}\,(\mathsf{Id}_{\mathcal{J}})$, in the sense that

$$(k \in \mathcal{J}) \qquad \alpha_k = F(\psi_k) : F(k) \xrightarrow{\sim} F(k).$$

- It then follows that the covariant isotropy group $\mathcal{Z} : \mathbf{Set}^{\mathcal{J}} \to \mathbf{Group}$ is *constant* on the group $\mathbf{Aut}(\mathsf{Id}_{\mathcal{J}})$ of natural automorphisms of $\mathsf{Id}_{\mathcal{J}} : \mathcal{J} \to \mathcal{J}$.

Presheaf Categories

- if $\mathcal{J}$ is a *rigid* category (i.e. has no non-trivial automorphisms), then the covariant isotropy group $\mathcal{Z} : \mathbf{Set}^{\mathcal{J}} \to \mathbf{Group}$ is constant on the trivial group.
- For any group G, the covariant isotropy group $\mathcal{Z} : \mathbf{Set}^{G} \to \mathbf{Group}$ of the category of G-sets is constant on the centre $Z(G)$ of the group G.
- More generally, for any monoid M, the covariant isotropy group $\mathcal{Z} : \mathbf{Set}^{M} \to \mathbf{Group}$ of the category of M-sets is constant on the group $\mathbf{Inv}(Z(M))$ of invertible elements of the centre of M.

Connections with Topos Theory

- If $\mathbb{T}$ is a quasi-equational theory, then $\mathbb{T}$ has a *classifying topos* $\mathcal{B}(\mathbb{T})$, which is a cocomplete topos that has a *universal model* of $\mathbb{T}$ and classifies all topos-theoretic models of $\mathbb{T}$ ([5], [6]).
- It has been shown that any Grothendieck topos $\mathcal{E}$ has a canonical internal group object called the *isotropy group* of the topos, which acts canonically on every object of the topos and formally generalizes the notion of conjugation ([2]).
- The covariant isotropy group $\mathcal{Z}_{\mathbb{T}}$ of a quasi-equational theory $\mathbb{T}$ *is* in fact the isotropy group object of the classifying topos $\mathcal{B}(\mathbb{T})$ of $\mathbb{T}$ ([2], [5]).

Conclusions

- Bergman's *element-free* characterization of the inner automorphisms of groups can be used to *define* inner automorphisms in arbitrary categories.
- We have extended Bergman's *logical* characterization of the (extended) inner automorphisms of groups, i.e. of the covariant isotropy group of $\mathbf{Group} = \mathbb{T}_{\mathbf{Grp}}\mathbf{mod}$, to the covariant isotropy group of $\mathbb{T}\mathbf{mod}$ for *any* quasi-equational theory $\mathbb{T}$: we have $\mathcal{Z}_{\mathbb{T}}(M) \cong \mathbf{DefInn}(M)$ for any $M \in \mathbb{T}\mathbf{mod}$.
- Using this characterization, we have obtained logical descriptions of the definable and (extended) inner automorphisms in **StrMonCat** and presheaf categories (among other algebraic categories).

Thank you!

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