Graphical Regular Logic

the complete 2D picture

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joint work with

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for

ACT 2021

About me:

PhD in mathematics at Johns Hopkins University

Expected graduation in 2022

Interested in: formal category theory, higher categories, homotopy type theory, computer implementations (and programming!)

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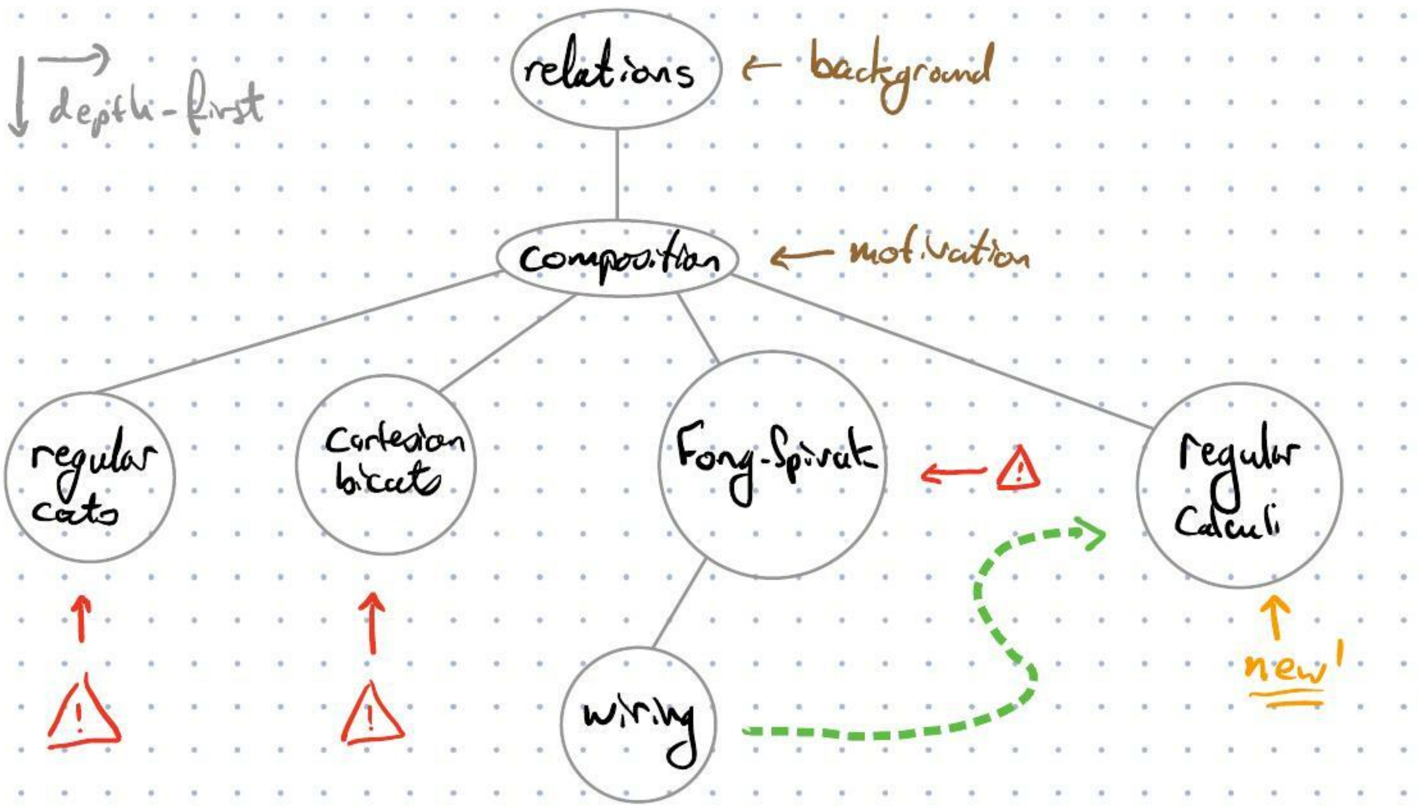
THE BIG PICTURE

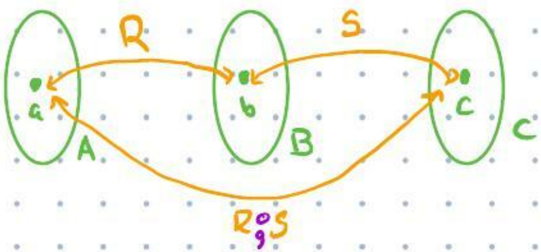


Relations

Definition: given sets $\{A_i\}_I$ a relation R is a subset of the product $R \subseteq \prod A_i$

Example: People $= \{\cdots\}$, Topics $= \{\cdots\}$, $R = \{$(you, functors), $\ldots\} \subseteq$ People $\times$ Topics

Definition: given relations $R \subseteq A \times B$ and $S \subseteq B \times C$ we may compose $R;S$ for a relation on $A \times C$



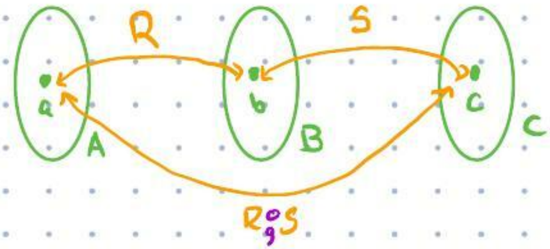
$R \subseteq$ People $\times$ Topics

$S \subseteq$ Topics $\times$ Talks@ACT21

$R;S =$ "those people who have exposure to a topic featured in an ACT21 talk"

Relations

<u>Definition</u>: given relations $R \subseteq A \times B$ and $S \subseteq B \times C$ we may **compose** $R ; S$ for a relation on $A \times C$



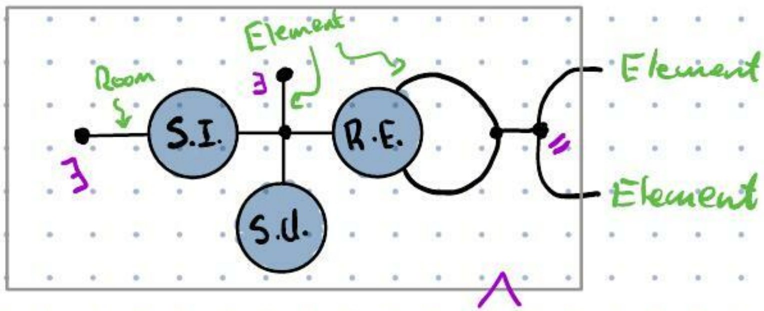
$$R ; S :\equiv \{ (a,c) \mid \exists b \, [(a,b) \in R \wedge (b,c) \in S] \}$$

Knowledge is often represented relationally

Element = {Sb, As, Al, Se, ...}   Room = {Cupboard under the stairs, ...}

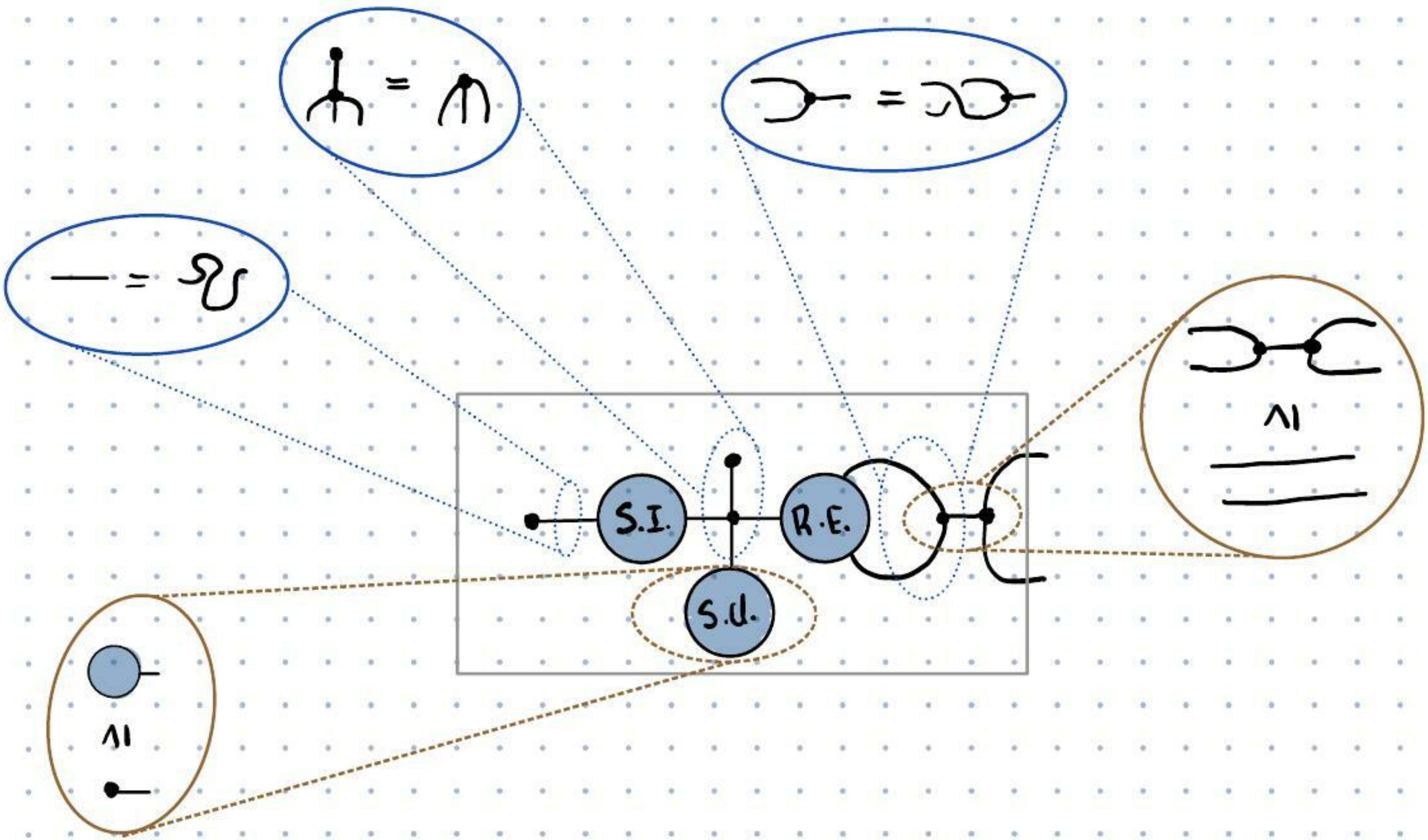
ReactEndo ⊆ Element$^{\times 3}$, StoredIn ⊆ Element × Room,

SafeforUndergrads ⊆ Element

Goal → 

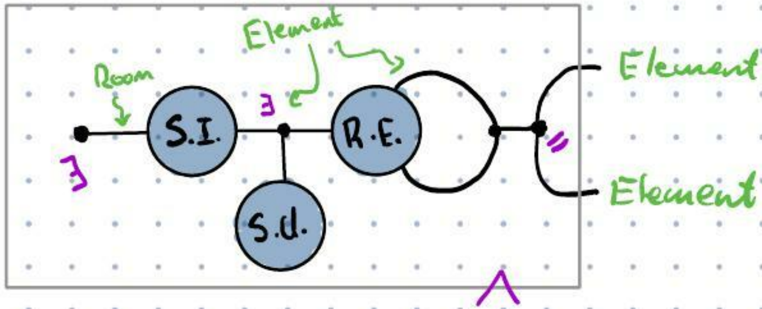
query ⇝ composition of the relations

Goal: Intuitive graphical manipulations



Element, Room; ReactEndo $\subseteq$ Element$^{\times 3}$, StoredIn $\subseteq$ Element $\times$ Room,
SafeForUndergrads $\subseteq$ Element

Goal →



vs

$$\text{"}\left\{ (e_1, e_2) \;\middle|\; \exists\, e_3, room \left[\begin{array}{l} e_1 = e_2 \wedge (e_1, e_2, e_3) \in \text{ReactEndo} \\ \wedge\, (e_3, room) \in \text{StoredIn} \\ \wedge\, e_3 \in \text{SafeForUndergrads} \end{array} \right] \right\}\text{"}$$

The theoretical minimum: (with apologies to Susskind & Hrabovsky)

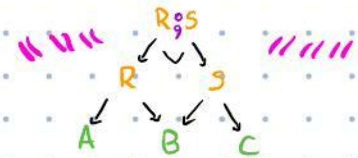
How much do we need to study relations and their compositions?

Def: Regular Logic is the fragment of first order logic generated by

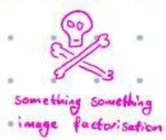
$\exists$	$=$	$\wedge$	true
existential quant.	equality	conjunction	true

$$\text{“} \left\{ (e_1, e_2) \;\middle|\; \exists e_3, room \begin{bmatrix} e_1 = e_2 \wedge (e_1, e_2, e_3) \in ReactEndo \\ \wedge\ (e_3, room) \in StoredIn \\ \wedge\ e_3 \in SafeForUndergrads \end{bmatrix} \right\} \text{”}$$

How do we study relations? Classical approach ⇝ regular categories

SET	$\xrightarrow{\text{generalise}}$ "Regular category" $\mathcal{R}$
$R \subseteq A \times B$	 object + jointly monic span ($R \rightarrowtail A \times B$)
$R ; S :\equiv \{(a,c) \mid \exists b\, [(a,b) \in R \wedge (b,c) \in S]\}$	

⚠



BUT

We want to privilege relations, and consider them between objects

⇝ morphisms!

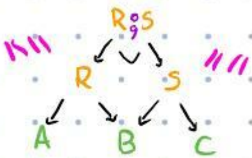
Carboni - Walters:

"Cartesian bicategories I"

See also "Categories, Allegories" by Freyd & Scedrov

Regular cat $\mathcal{R}$ we form a bicategory $\mathbb{R}\mathrm{el}\,\mathcal{R}$,

objects •
morphisms • ← •
2-morphs

- objects $ob\mathcal{R}$
- $\mathbb{R}\mathrm{el}\,\mathcal{R}(A,B) :\equiv$ relations $A \leftarrow R \rightarrow B$
  jointly monic spans
- Composition 

- $\mathbb{R}\mathrm{el}\,\mathcal{R}(A,B)$ are posets, $R \subseteq S$
  $\Rightarrow$ "po-category"

category "enriched" over posets

bicategory "thin" in dim. 2

- monoidal
  ($A \times B$ in $\mathcal{R}$)
- other structure

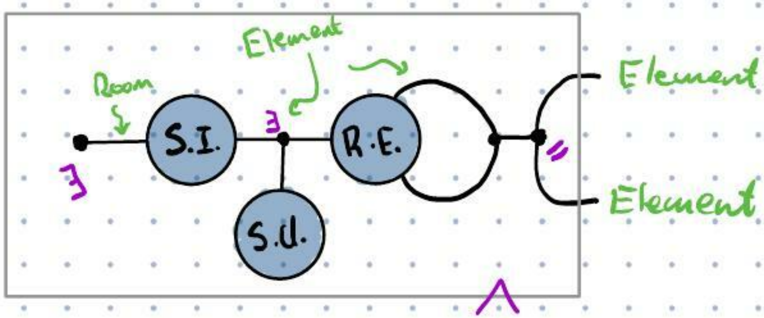
Carboni - Walters:

"Cartesian bicategories I"

symmetric monoidal po-categories + structure + properties do work

⚠ BUT ⚠

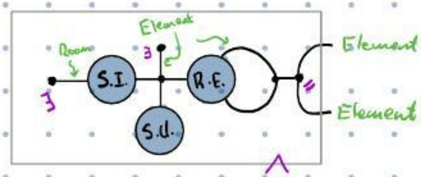
there are no pictures



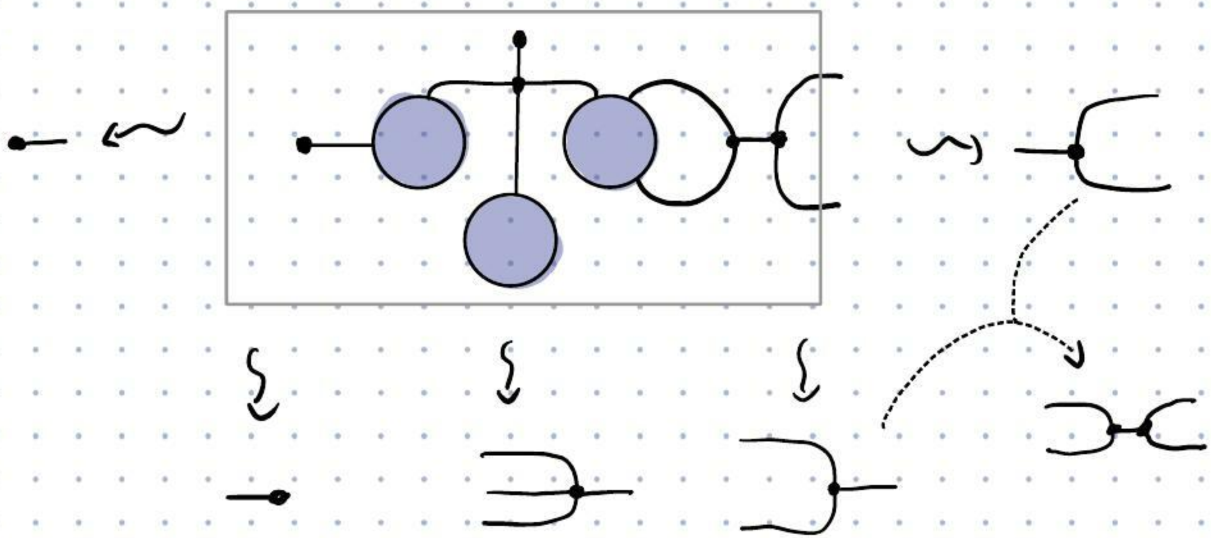
Fong - Spivak:

"Regular and relational categories: revisiting 'Cartesian bicategories I'"

Let's look at



abstractly, notice the wiring:



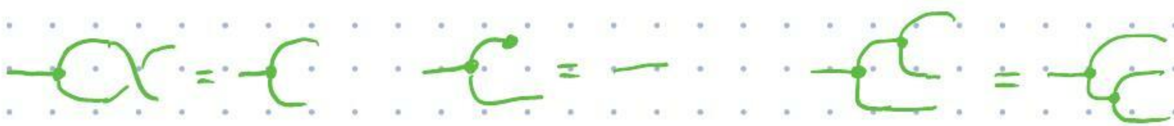
Fong - Spivak: "Regular and relational categories: revisiting 'Cartesian bicategories I' "

Def: The "po-prop" for wiring, $\mathbb{W}$, is the symmetric monoidal po-category generated by pictures

morphisms [

variable duplication/equality $+\exists$

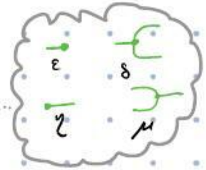
and laws



. . .  . . .

constraining variables to be equal is less general

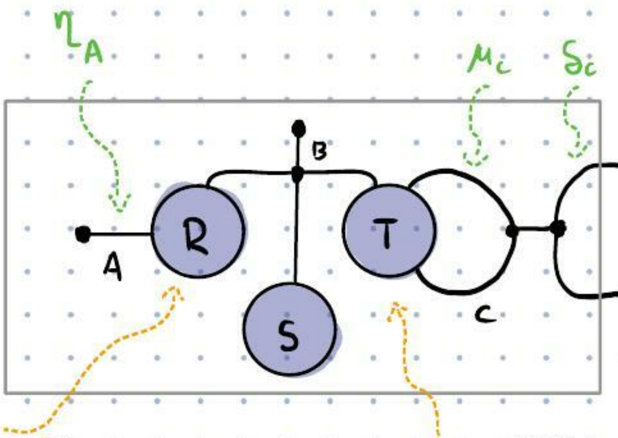
Fong - Spivak: "Regular and relational categories: revisiting 'Cartesian bicategories I' "

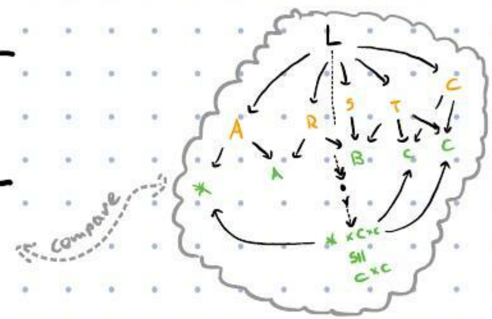


Category + homs are posets + ...

Def: A supply for $\mathbb{W}$ in a symmetric monoidal po-category $\mathbb{C}$ is a way to draw string/wiring diagrams for the morphisms of $\mathbb{C}$ where the wires are controlled by $\mathbb{W}$

wires of $\mathbb{W}$ [





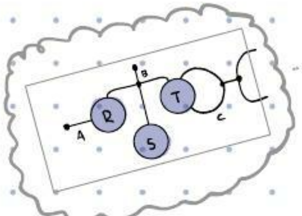
morph of $\mathbb{C}$ "relations" [$R \in \mathbb{C}(A,B)$ $T \in \mathbb{C}(B, C \otimes C)$

Fong - Spivak : "Regular and relational categories: revisiting 'Cartesian bicategories I' "

as 2-categories

Thm:

$$\left\{\begin{array}{l}\text{symm. monoidal po-cat} \\ \quad + \text{supply } \mathbb{W} \\ \quad + \underline{\qquad\qquad}\end{array}\right\} \xleftarrow[\mathbb{R}\mathrm{el}]{\simeq} \left\{\text{regular categories}\right\}$$



Moral

a po-category behaves like

"objects + relations between them"

precisely when we can draw $\mathbb{W}$ wiring pictures

(+ some stuff)

Domain of objection:

Fong-Spivak supply of $\mathbb{B}\mathbb{W}$ wiring does give pictures **BUT** technology is still a symmetric monoidal po-category

⇝ relations are morphisms $R \in \mathbb{C}(A, B)$ ⇝ dom & cod ⚠

⇝ we are privileging binary relations in a sense,

R on $A \otimes B \otimes C$ must be encoded in $\begin{cases} \mathbb{C}(A, B \otimes C) \\ \text{or } \mathbb{C}(A \otimes B, C) \\ \text{or } \ldots \end{cases}$

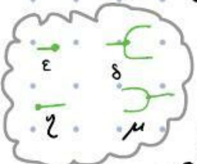
and it is unnatural to decide!

Regular calculi: (new & improved!)

A, B are contexts
$\Rightarrow$ $A \otimes B$ is a context

Def: A regular calculus $(\mathbb{C}, P)$ is

1) symmetric monoidal po-category $\mathbb{C}$ .... $\begin{cases} \text{ob } \mathbb{C} = \text{"contexts"} \\ \text{mor } \mathbb{C} = \text{"context maps"} \end{cases}$



2) supply of $\mathbb{W}$ in $\mathbb{C}$ .... pictures + structured contexts ($\exists$, dup, $=$, ...)

3) "ajax" po-functor $P : \mathbb{C} \to \mathbb{P}\text{oset}$ .... predicates, each $P(\Gamma)$ has $\wedge$ and true

lax monoidal + laxators are right adjoints

$\varphi \in P(A \otimes B)$

$\varphi(a,b) :\equiv (a,b) \in R$ ($R \subseteq A \times B$ / φ formula in A, B) $\{(a,b) \mid \varphi(a,b)\}$

Example:

$\mathcal{R}$ a regular category (e.g. Set) $\rightsquigarrow$ $\mathbb{P}\mathrm{rd}\,\mathcal{R}$ a regular calculus:

we need
1) a symmetric monoidal po-cat ← $\mathbb{R}\mathrm{el}\,\mathcal{R}$
2) a supply of $\mathbb{W}$ in $\mathbb{R}\mathrm{el}\,\mathcal{R}$ ← theorem of Fong-Spivak
3) ajax po-functor $\mathbb{R}\mathrm{el}\,\mathcal{R} \to \mathbb{P}\mathrm{oset}$ ← representable on 1, $\mathbb{R}\mathrm{el}\,\mathcal{R}(\underline{1}, -)$

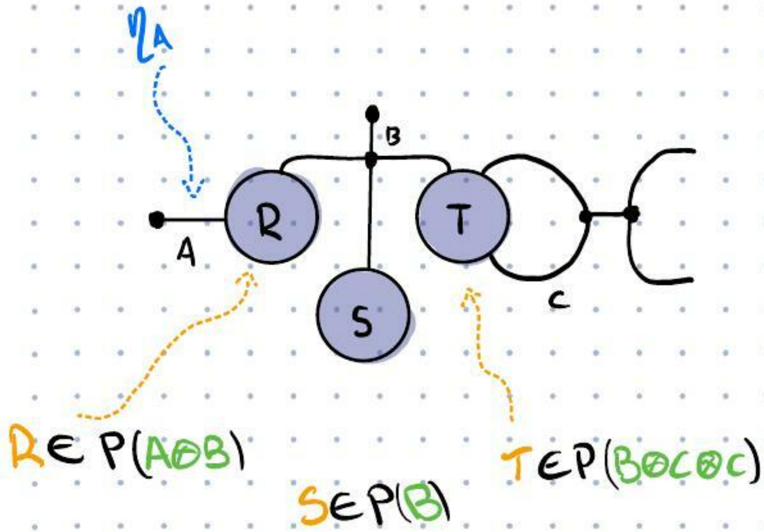
$\Rightarrow$

- contexts are objects of $\mathcal{R}$
- predicates in context Γ are subobjects of Γ!
  eg: $\Gamma = A \times B \times C$, $P(\Gamma) = \{R \subseteq A \times B \times C\}$ — no binary privilege!
- $\exists$, $\wedge$, $=$, true, ... are all exactly correct!

Graphical regular logic

$(\mathbb{C}, P)$ regular calculus $\Rightarrow$ supply of $\mathbb{W}$ wiring in $\mathbb{C}$ $\Rightarrow$ pictures!

Def: In a regular calculus $(\mathbb{C}, P)$, a graphical term is a wiring diagram in $\mathbb{C}$ where the internal wires are annotated by predicates:



represents the predicate in $P(C \otimes C)$ obtained by using $\eta_A, \delta_B, \mu_C, \delta_C, \ldots$ on R, S, and T

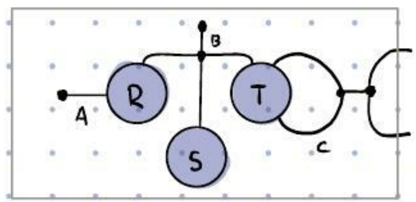
Regular calculi vs regular categories:

Proposition (C.-Fong-Spivak):

$\mathbb{P}\mathrm{rd} : \{\text{regular cats}\} \longrightarrow \{\text{regular calculi}\}$ is a 2-functor

Theorem (CFS): $\mathbb{P}\mathrm{rd}$ has a left "bi-adjoint" Syn, the syntactic po-category construction

given by graphical terms



but that's not all...

<u>Thm (CFS)</u>: The counit of the bi-adjunction

$$\{\text{regular cats}\} \underset{\mathbb{P}\mathrm{rd}}{\overset{\mathrm{Syn}}{\rightleftarrows}} \{\text{regular calculi}\} \quad (\perp_{bi})$$

is an adjoint equivalence.

$\rightsquigarrow$ $\mathbb{P}\mathrm{rd}$ is an "embedding"!

⤳ we can work graphically in any regular category!

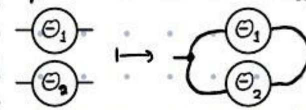
as regular cats!

for any regular cat $\mathcal{R}$,

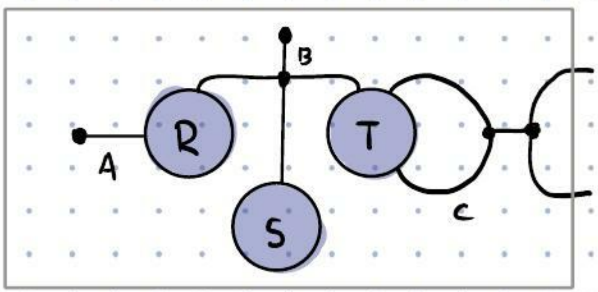
$\mathrm{SynPrd}\,\mathcal{R} \simeq \mathcal{R}$

objects

equaliser:



Goal →



Thanks!

Carboni - Walters "Cartesian bicategories I"

Fong - Spivak
- "Hypergraph categories"
- "Supplying bells and whistles in monoidal categories"
- "Regular and relational categories, revisiting 'Cartesian bicategories I'"

Clingman - Fong - Spivak "Graphical regular calculus I (& II)" [forthcoming!]

What about the other paper?

There was a previous attempt by Fong-Spivak at graphical regular logic,

but it was not high-enough dimensional to be correct:

$\rightsquigarrow$ instead of general regular calculi, only free reg. cats.

- not rich enough
- inherently low dimensional (set vs "3-cat" $\mathbb{C}$)

$\rightsquigarrow$ instead of bi-adjunction (3d statement!), only adjunction in 1d