The word problem for braided monoidal categories is unknot-hard

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Applied Category Theory 2021

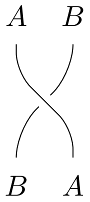
Periodic table of n-categories

$k \backslash n$	0	1	2	3
0	set	category	2-category	3-category
1	$\{\star\}$	monoid	monoidal cat.	monoidal 2-cat.
2	$\vdots$	$\{\star\}$	comm. monoid	braided monoidal cat.
3		$\vdots$	$\{\star\}$	comm. monoid
4			$\vdots$	$\{\star\}$

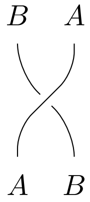
Braided monoidal categories

Definition

A ***braided monoidal category*** $\mathcal{C}$ *is a monoidal category equipped with a natural isomorphism* $\sigma_{A,B} : A \otimes B \to B \otimes A$ *satisfying the hexagon equations.*



(a) String diagram for $\sigma_{A,B}$



(b) String diagram for $\sigma_{A,B}^{-1}$

Axioms of braided monoidal categories

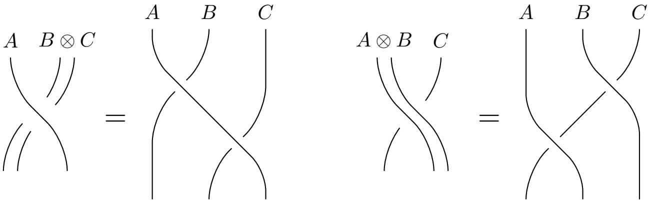
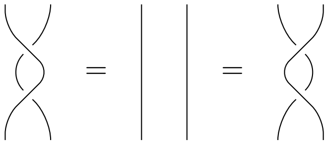
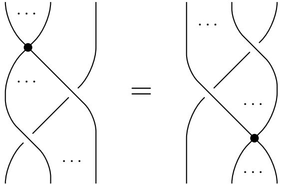


Figure: Hexagon equations

Axioms of braided monoidal categories



(a) Reidemeister 2 move (σ is an iso)

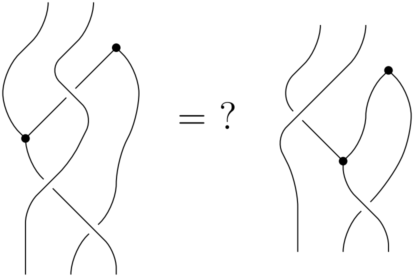


(b) Pull-through move (naturality of σ)

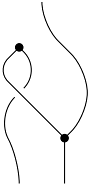
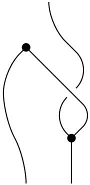
Word problem for braided monoidal categories

Decision problem

Given two expressions of morphisms in a free braided monoidal category, determine if the morphisms they represent are equal.



A few examples


$= ?$


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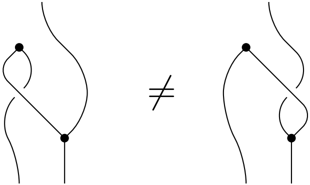
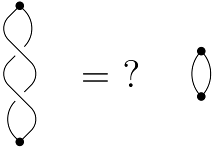
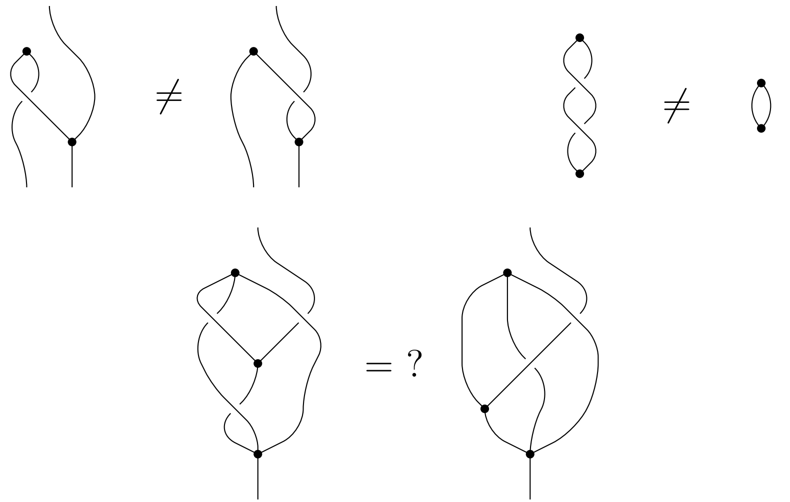
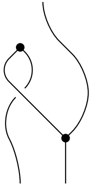




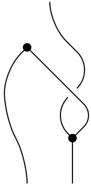
A few examples



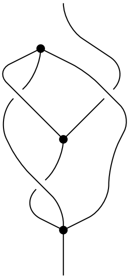
A few examples



$\neq$



$\neq$



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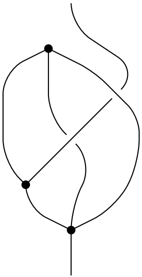
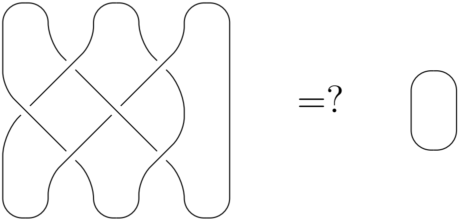
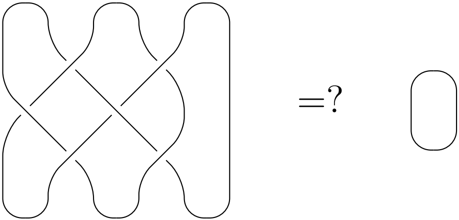


Unknotting knots: a well-studied problem

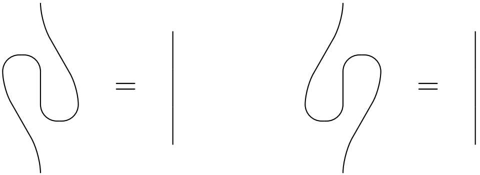


Unknotting knots: a well-studied problem

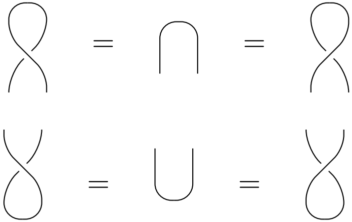


This decision problem is known to be in **NP** and **coNP**, but no polynomial time algorithm is known for it.

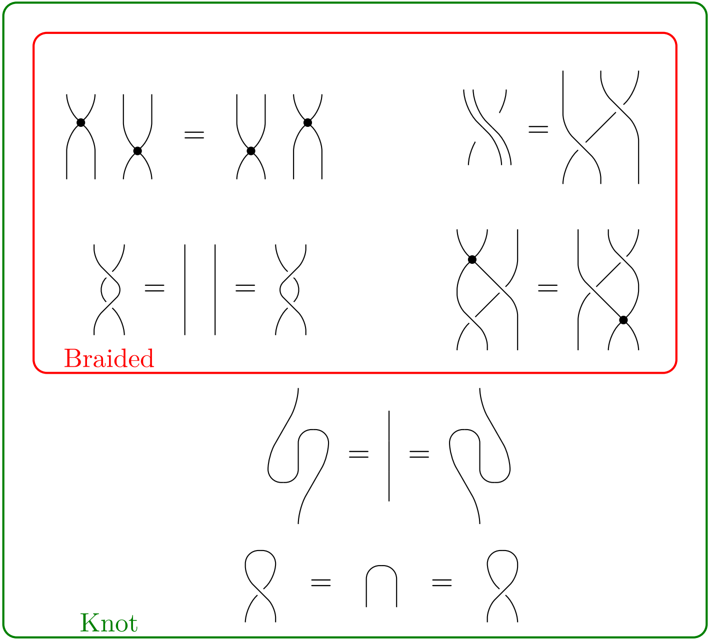
Extra equations for knots



(a) Yanking moves



(b) Reidemeister type 1 moves

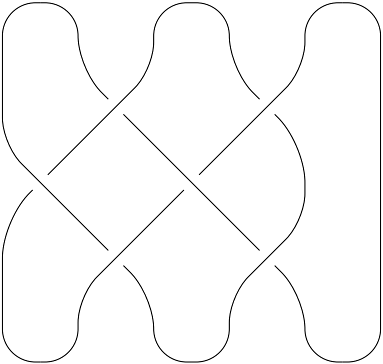

Braided
Knot

Our result

Theorem

The unknotting problem can be polynomially reduced to the word problem for braided monoidal categories.

First attempt



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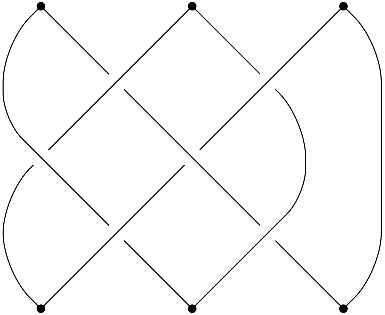

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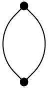
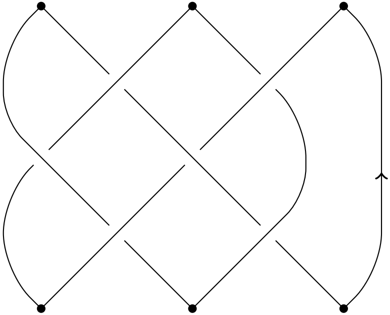


First attempt



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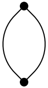
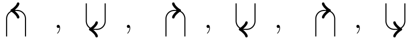
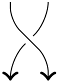
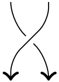


Cap-cup cycle: ()

Writhe

Definition

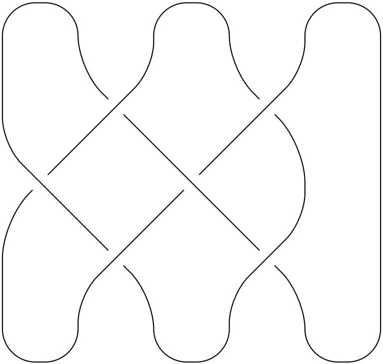
*Given an oriented knot diagram, its **writhe** is obtained by summing the local writhe at each crossing:*

$w($  $) = +1$ $\qquad$ $w($  $) = -1$

Lemma

The axioms of braided monoidal categories preserve the writhe.

Idea of the proof



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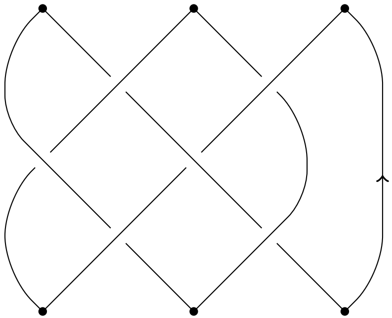

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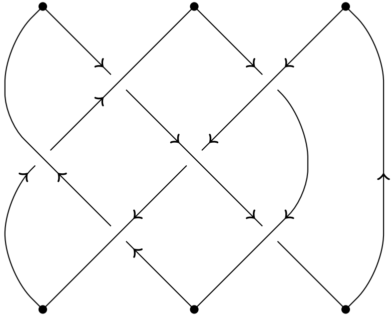

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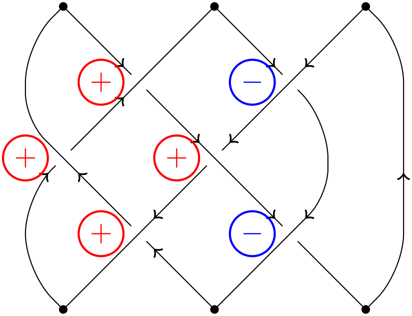

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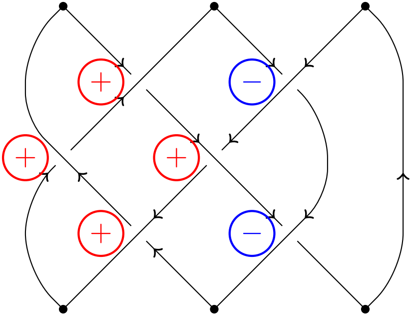
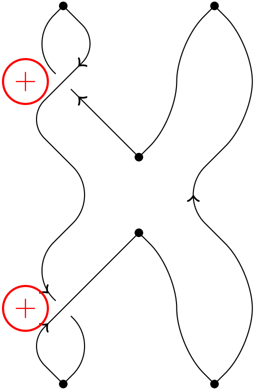


$=_{\mathrm{BMC}}?$



Conclusion

The word problem for braided monoidal categories is at least as hard as the unknotting problem.

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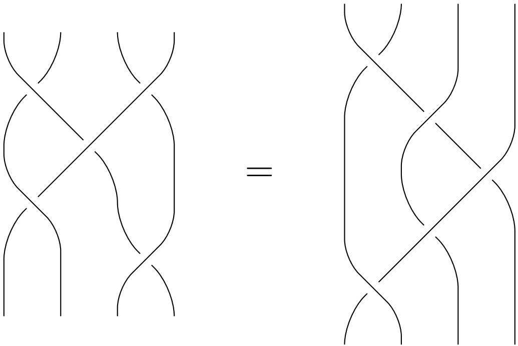
Is the word problem for braided monoidal categories even **decidable**?

Braids

The free braided monoidal category on a single object is the category of braids.



Word problem for braids

Theorem

The word problem for braids can be solved in quadratic time in the length of the braids.

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But braided monoidal categories can have non-braid morphisms!