

A Categorical Semantics for Bounded Petri Nets

Fabrizio Genovese

0000-0001-7792-1375

University of Pisa & Statebox

fabrizio.romano.genovese@gmail.com

Fosco Loregian

0000-0003-3052-465X

Tallinn University of Technology

fosco.loregian@gmail.com

Daniele Palombi

0000-0002-8107-5439

Sapienza University of Rome

danielepalombi@protonmail.com

We provide a categorical semantics for bounded Petri nets, both in the collective- and individual-token philosophy. In both cases, we describe the process of bounding a net internally, by just constructing new categories of executions of a net using comonads, and externally, using lax-monoidal-lax functors. Our external semantics is non-local, meaning that tokens are endowed with properties that say something about the global state of the net. We then prove, in both cases, that the internal and external constructions are equivalent, by using machinery built on top of the Grothendieck construction. The individual-token case is harder, as it requires a more explicit reliance on abstract methods.

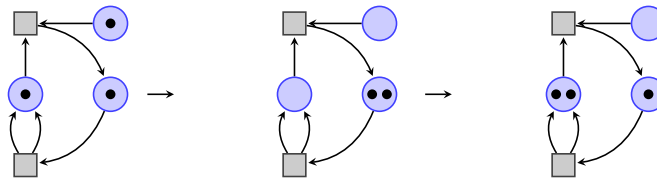
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A video presentation of this paper can be found on Youtube at [mv4dyTNgz60](#).

1 Introduction

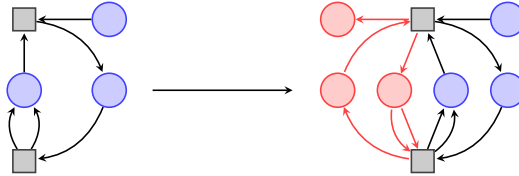
A Petri net is a simple thing: it consists of *places* – depicted as circles – and *transitions* – depicted as squares. Places are seen as holding resources, called *tokens* and represented by black circles, while transitions are seen as processes that convert a pre-determined amount of some resource into a pre-determined amount of some other resource. The action of transitions is considered atomic, and such an event is called *firing*. Sequences of firings are also called *executions*.



This innocent definition [16] has proven very useful in domains ranging from concurrency [11, 18], where it is used to represent resource flow in distributed computing, to chemistry [17, 3], where it models chemical reactions. In both cases, it becomes important to characterize which sequences of transitions carry a given distribution of tokens for the net – called *marking* – to some other marking. Here is where the similitude between applications specializes: for instance, in chemistry, tokens represent molecules/atoms of a given type specified by the place in which they live. Two tokens in the same place are physically indistinguishable, and we do not care which token is consumed by a fixed transition. In computer science applications – e.g. [21] – tokens are instead seen as terms of a given type, and it is important to distinguish

between them. These two philosophies are called *collective-token* and *individual-token* philosophy, respectively [22].

In any case, one other important question is to establish when a given Petri net is *bounded*, meaning that starting from a given marking, no place will hold more than a pre-determined number of tokens throughout any possible firing [17]. Traditionally, there is a simple way of turning any net into a bounded one: we double-up the places adding what we call *anti-places* (depicted in red), and edit transitions so that each input (output) from (to) a place is now paired with a corresponding output (input) to (from) the corresponding anti-place:



It is easy to see that for any marking, the amount of tokens in each anti-place will determine the maximum number of tokens the corresponding place can hold through any execution, and vice-versa.

Building on the long-standing tradition of looking at Petri nets through a categorical lens [14], we want to give a category-theoretic interpretation of this process, both in the collective-token philosophy and in the individual-token one. In each case, we will provide two different categorical ways to bound a net, which we will call *internal* and *external*, respectively. Then, we will prove that internal and external bounding are conceptually the same thing, generalizing a technique already established in [10]. Notably, in doing so, we will employ a new kind of categorical semantics for Petri nets, which we call *non-local* [9], where tokens come endowed with information concerning the global properties of the net itself.

2 Petri nets and their executions

Notation 1. Let S be a set; denote with S^\oplus the set of multisets over S . Multiset sum will be denoted with \oplus , and difference (only partially defined) with \ominus . S^\oplus with \oplus and the empty multiset is isomorphic to the free commutative monoid on S .

Definition 1 (Petri net). A Petri net is a couple functions $T \xrightarrow{s,t} S^\oplus$ for some sets T and S , called the set of places and transitions of the net, respectively.

Definition 2 (Markings and firings). A marking for a net $T \xrightarrow{s,t} S^\oplus$ is an element of S^\oplus , representing a distribution of tokens in the net places. A transition u is enabled in a marking M if $M \ominus s(u)$ is defined. An enabled transition can fire, moving tokens in the net. Firing is considered an atomic event, and the marking resulting from firing u in M is $M \ominus s(u) \oplus t(u)$.

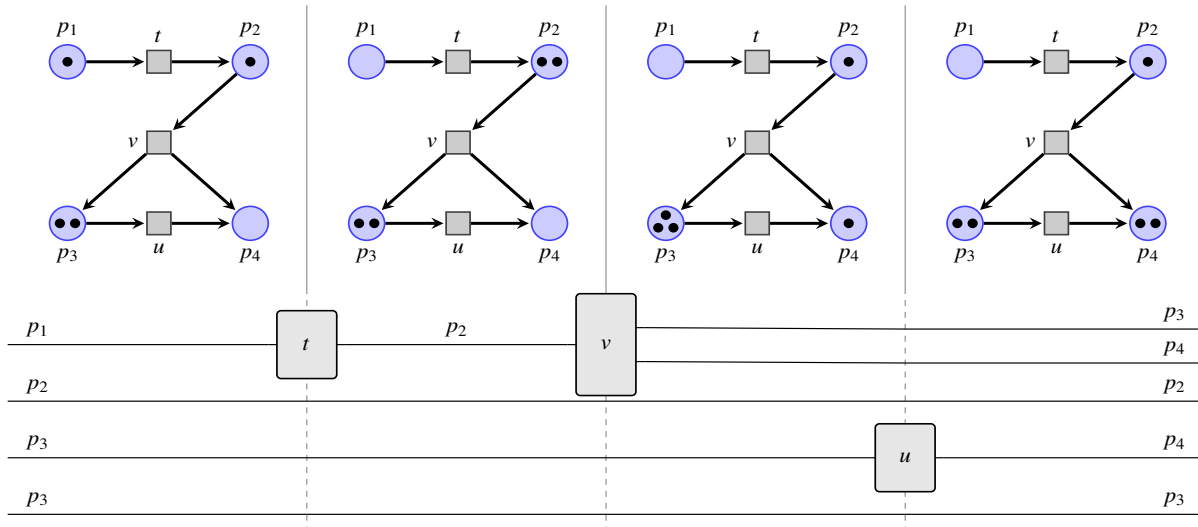
As we said, depending if we want to consider tokens as indistinguishable or not, we can interpret nets under two different philosophies, traditionally called collective-token and individual-token, respectively. Category theory is helpful in pinpointing precisely the meaning of this distinction by providing different notions for the *category of executions of a given net*. In the collective-token philosophy, the executions of a Petri net are taken to be morphisms in a *commutative monoidal category* – a category whose monoid of objects is commutative:

Definition 3 (Category of executions – collective-token philosophy). Let $N : T \xrightarrow{s,t} S^\oplus$ be a Petri net. We can generate a free commutative strict monoidal category (FCSMC), $\mathfrak{C}(N)$, as follows:

- The monoid of objects is S^\oplus . Monoidal product of objects A, B , denoted with $A \oplus B$, is given by the multiset sum.
- Morphisms are generated by T : each $u \in T$ corresponds to a morphism generator $su \xrightarrow{u} tu$; morphisms are obtained by considering all the formal (monoidal) compositions of generators and identities.

A detailed description of this construction can be found in [13].

Example 1. In Definition 3 objects represent markings of a net: $A \oplus A \oplus B$ means “two tokens in A and one token in B ”. Morphisms represent executions of a net, mapping markings to markings. A marking is reachable from another one if and only if there is a morphism between them.



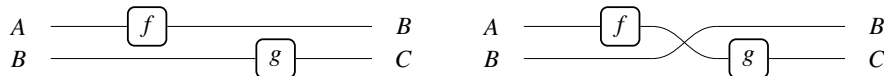
As for the individual-token philosophy, we obtain a suitable semantics by relaxing the commutativity requirement in Definition 3.

Definition 4 (Category of executions – individual-token philosophy). Let $N : T \xrightarrow{s,t} S^\oplus$ be a Petri net. We can generate a free symmetric strict monoidal category (FSSMC), $\mathfrak{F}(N)$, as follows:

- The monoid of objects is S^\otimes , the set of strings over S . Monoidal product of objects A, B , denoted $A \otimes B$, is given by string concatenation.
- Morphisms are generated by T : each $u \in T$ corresponds to a morphism generator $su \xrightarrow{u} tu$, where su, tu are obtained by choosing some ordering on their underlying multisets; morphisms are obtained by considering all the formal horizontal and vertical compositions of generators, identities and symmetries.

A detailed description of this construction can be found in [7].

Example 2. The interpretation of Definition 4 is as in Definition 3, but now switching tokens around is not anymore a trivial operation. For instance, the following morphisms are considered different in Definition 4, and equal in Definition 3:



A great deal of work has been devoted to understanding how categories of Petri nets and categories of (commutative, symmetric) strict monoidal categories are related [14, 19, 8, 2, 4, 6, 1]. We won't focus on these issues in this paper; instead, we will be interested in recovering the idea of "bounding a net" as a functor between the category of executions of a net and some other category.

The approach we pursue will be very similar to the one we already adopted for coloured nets in [10], with the difference that our functors will have to be lax-monoidal-lax, something we already partially exploited in [9]. For obvious reasons, we start from the collective-token case, which is simpler.

3 Bound semantics in the collective-token philosophy

We briefly described the process of turning a Petri net into a bounded one in Section 1. Categorically, this process can be implemented in the executions semantics, as follows.

Definition 5 (Internal bound semantics – collective-token philosophy). *Let N be a Petri net, and consider $\mathfrak{C}(N)$, its corresponding FCSMC. The internal bound semantics of N in the collective-token philosophy is given by the FCSMC $\mathfrak{C}_B(N)$ generated as follows:*

- For each generating object A in $\mathfrak{C}(N)$, we introduce object generators A^+ , A^- .
- For each generating morphism

$$\bigoplus_{i=1}^n A_i \xrightarrow{u} \bigoplus_{j=1}^m B_j$$

in $\mathfrak{C}(N)$, we introduce a morphism generator:

$$\bigoplus_{i=1}^{m \vee n} (A_i^+ \oplus B_i^-) \xrightarrow{u} \bigoplus_{i=1}^{m \vee n} (A_i^- \oplus B_i^+)$$

where we adopt the convention that, if $n < m$ (respectively $m > n$), then A_{n+1}, \dots, A_m (respectively B_{m_1}, \dots, B_n) are taken to be equal to I , the monoidal unit, and $m \vee n := \max\{m, n\}$.

Example 3. Notice that by definition, every FCSMC is presented by a Petri net. Following Definition 5, it is easy to see that if we consider N to be the net in Section 1, then $\mathfrak{C}_B(N)$ is presented exactly by the bounded net we naïvely associated to it. Indeed, generating objects of type A^+ represent a token in the place A of the net while generating objects of type A^- represent tokens in the corresponding anti-place. Generating morphisms of $\mathfrak{C}_B(N)$ have inputs and outputs exactly as the transitions in the net obtained by bounding N .

Proposition 1. The assignment $\mathfrak{C}(N) \mapsto \mathfrak{C}_B(N)$ defines a comonad in the category of FCSMCs and strict monoidal functors between them, **FCSMC**.

Proof. First of all, we have to prove that the procedure is functorial. For any strict monoidal functor $F : \mathfrak{C}(N) \rightarrow \mathfrak{C}(M)$ we define the action on morphisms $\mathfrak{C}_B(F) : \mathfrak{C}_B(N) \rightarrow \mathfrak{C}_B(M)$ as the following monoidal functor:

- If for a generating object A of $\mathfrak{C}(N)$ it is $FA = B$, then $\mathfrak{C}_B(F)A^+ = B^+$ and $\mathfrak{C}_B(F)A^- = B^-$.
- If for a generating morphism t of $\mathfrak{C}(N)$ it is $Ft = u$, then $\mathfrak{C}_B(F)t = u$.

Identities and compositions are clearly respected, making $\mathfrak{C}_B(-)$ an endofunctor in **FCSMC**. As a counit, on each component N we define the strict monoidal functor $\varepsilon_N : \mathfrak{C}_B(N) \rightarrow \mathfrak{C}(N)$ sending:

- The generating object A^+ to A , and the generating object A^- to I , the monoidal unit of $\mathfrak{C}(N)$.

- A generating morphism

$$\bigoplus_{i=1}^{m \vee n} (A_i^+ \oplus B_i^-) \xrightarrow{u} \bigoplus_{i=1}^{m \vee n} (A_i^- \oplus B_i^+)$$

To the generating morphism

$$\bigoplus_{i=1}^n A_i \xrightarrow{u} \bigoplus_{j=1}^m B_j$$

The procedure is natural in the choice of N , making ε into a natural transformation $\mathfrak{C}_B(-) \rightarrow \text{id}_{\mathbf{FCSMC}}$.

As for the comultiplication, on each component N we define the strict monoidal functor $\delta_N : \mathfrak{C}_B(N) \rightarrow \mathfrak{C}_B(\mathfrak{C}_B(N))$ sending:

- The generating object A^+ is sent to the generating object $A^{++} \oplus A^=$.
- The generating object A^- is sent to the generating object $A^{\mp} \oplus A^{\pm}$.
- A generating morphism of $\mathfrak{C}_B(N)$

$$\bigoplus_{i=1}^{m \vee n} (A_i^+ \oplus B_i^-) \xrightarrow{u} \bigoplus_{i=1}^{m \vee n} (A_i^- \oplus B_i^+)$$

Is sent to the generating morphism $\mathfrak{C}_B(\mathfrak{C}_B(N))$

$$\bigoplus_{i=1}^{m \vee n} (A_i^{++} \oplus A_i^= \oplus B_i^{\pm} \oplus B_i^{\mp}) \xrightarrow{u} \bigoplus_{i=1}^{m \vee n} (A_i^{\mp} \oplus A_i^{\pm} \oplus B_i^= \oplus B_i^{++})$$

The naturality of δ , its coassociativity and the fact that ε is a counit are all straightforward checks. \square

In the spirit of [10], our goal is now to frame this comonad into a bigger context, one of Petri nets with a *semantics* attached to them. A semantics for a Petri net is just a functor from its category of executions (modulo choice of token philosophy) to some other monoidal category \mathcal{S} .

In [10], this functor was required to be strict monoidal. This backed up the interpretation that a semantics “attaches extra information to tokens”, which is then used by the transitions in some way.

In here, we require this functor to be *lax-monoidal-lax*. Laxity amounts to say that we can attach *non-local* information to tokens: they may “know” something about the overall state of the net, and the laxator represents the process of “tokens joining knowledge”. If this sounds handwavy, we guarantee it will be made clear shortly.

Definition 6 (Non-local semantics – collective-token philosophy). *Let N be a Petri net and let \mathcal{S} be a monoidal bicategory [20]. A Petri net with a commutative non-local semantics is a couple (N, N^\sharp) , with N^\sharp a lax-monoidal-lax functor $\mathfrak{C}(N) \rightarrow \mathcal{S}$.*

*A morphism $(N, N^\sharp) \rightarrow (M, M^\sharp)$ of Petri nets with commutative semantics is a strict monoidal functor $\mathfrak{C}(N) \xrightarrow{F} \mathfrak{C}(M)$ making the obvious triangle commute. We denote the category of Petri nets with a non-local commutative semantics with **Petri** $^{\mathcal{S}}$.*

We now show how we can encode the information of a net being bounded as some particular kind of non-local semantics.

Notation 2. *Let **Span** be the monoidal bicategory of sets, spans and span morphisms. A morphism $A \rightarrow B$ in **Span** consists of a set S and a pair of functions $A \leftarrow S \rightarrow B$. When we need to notationally extract this information from f , we write $A \xleftarrow{f_1} S_f \xrightarrow{f_2} B$. We sometimes consider a span as a morphism $f : S_f \rightarrow A \times B$, thus we may write $f(s) = (a, b)$ for $s \in S_f$ with $f_1(s) = a$ and $f_2(s) = b$. Recall moreover that a 2-cell in **Span** $f \Rightarrow g$ is a function $\theta : S_f \rightarrow S_g$ such that $f = \theta \circ g$.*

Remark 1. In a free commutative (resp. symmetric) strict monoidal category, there are no equations between generators. As such, even if a morphism can be decomposed in a composition of tensors of generators (and symmetries) in multiple ways, the generators used – as well as how many times they are used – is an invariant of this decomposition. Given this, there exists a well-defined function

$$\chi : \text{Hom}_{\mathfrak{F}(N)} \longrightarrow \left\{ \begin{array}{l} \text{generator mor-} \\ \text{phisms of } \mathfrak{F}(N)^{\oplus} \end{array} \right\}$$

mapping each morphism f to a multiset counting how many times each morphism generator is used in the decomposition of f .

Definition 7 (External bound semantics – collective-token philosophy). Given a Petri net $N : T \xrightarrow{s,t} S^{\oplus}$, define the following lax-monoidal-lax functor $N^{\sharp} : \mathfrak{C}(N) \rightarrow \mathbf{Span}$:

- Each object A of $\mathfrak{C}(N)$ is mapped to the set S^{\oplus} , the set of objects of $\mathfrak{C}(N)$.
- Each morphism $A \xrightarrow{f} B$ is sent to the span:

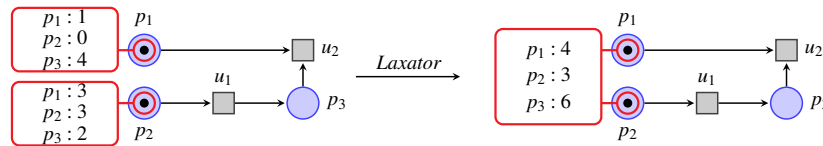
$$S^{\oplus} \xleftarrow{t} \chi(f)^{-1} \xrightarrow{s} S^{\oplus}$$

with s, t denoting source and target, respectively, of a morphism in $\mathfrak{C}(N)$.

Proposition 2. The functor of Definition 7 is lax-monoidal-lax. Functors as in Definition 7 form a subcategory of $\mathbf{Petri}^{\mathbf{Span}}$, which we call $\mathbf{Petri}_B^{\mathbf{Span}}$.

Proof. Since $\chi(\text{id}_A)$ is the empty multiset, and identities and symmetries coincide, $\chi(\text{id}_A)^{-1}$ coincides with the set of objects of $\mathfrak{C}(N)$, and identities are preserved strictly. As for composition, the laxating 2-cell is given by the obvious inclusion $\chi(f)^{-1} \times_{S^{\oplus}} \chi(g)^{-1} \rightarrow \chi(f \circ g)^{-1}$ obtained by composition. A very similar argument¹ holds for the monoidal product noticing that $\chi(f \circledast g) = \chi(f \oplus g)$. The coherence conditions are tedious to check but straightforward, given that the structure of the laxators is very simple. \square

Example 4. We now try to shed some light on Definition 7. Consider the net below. It has three places, which according to Definition 7 are all sent to the set of multisets over themselves. We interpret tokens as endowed with elements of such a set. These represent markings on the anti-places of the net. In our example, the token in p_1 “knows” that there is one token in the anti-place p_1^- and four tokens in the anti-place p_3^- , while p_2 “knows” that there are three tokens in p_1^- and p_2^- , two in p_3^- . The laxator allows considering these two tokens as a unique entity, with their respective “local pieces of knowledge” summing up.



Transitions u_1, u_2 generate the morphisms of the category of executions of the net. Looking at our definition, u_1 is mapped to the span function that subtracts p_3 to the multiset in input and adds p_2 to it. This represents the flow of anti-tokens, which goes in the opposite way wrt the flow of tokens of u_1 . This again backs up our intuition since anti-places are wired to transitions in the opposite way of their corresponding places.

¹If this seems very strange, recall Example 2. As objects commute, morphisms in $\mathfrak{C}(C)$ obey a very weak causal flow, and composition and monoidal product are quite similar.

3.1 Internalization

As of now, we defined two different kinds of bound semantics for Petri nets in the collective-token philosophy, specified by Definition 5 and Definition 7. The former is labelled *internal* because places and anti-places are all put together in the same category, while the latter is labelled *external* since the information about bounded tokens is encoded in a functor. We now show that the two approaches are one and the same.

Theorem 1. *Let (N, N^\sharp) be an object of $\mathbf{Petri}_B^{\mathbf{Span}}$. The category $\mathfrak{C}_B(N)$ of Definition 5 is isomorphic to the category $\int N^\sharp$ defined as follows:*

- *Objects of $\int N^\sharp$ are couples (X, x) where X is a object of \mathcal{C} and $x \in N^\sharp X$.*
- *Morphisms $(X, x) \rightarrow (Y, y)$ of $\int N^\sharp$ are couples (f, s) with $f : X \rightarrow Y$ a morphism of \mathcal{C} and $s \in S_{N^\sharp f}$ such that $N^\sharp f(s) = (x, y)$.*

Proof. Leveraging on the equivalence of categories:

$$\Gamma : \mathbf{Cat}/\mathfrak{C}(N) \simeq \mathbf{Cat}_l[\mathfrak{C}(N), \mathbf{Span}] : \int$$

(more details in Definitions 10 and 11, and Theorem 2), it is sufficient to consider the counit $\varepsilon_N : \mathfrak{C}_B(N) \rightarrow \mathfrak{C}(N)$ of Proposition 1, and to notice that $\Gamma \varepsilon_N = N^\sharp$. Then Theorem 2 guarantees that $\int \Gamma \varepsilon_N \simeq \varepsilon_N$, from which the thesis follows. \square

4 Bound semantics in the individual-token philosophy

We now want to generalize the results of Section 3 to the individual-token philosophy. In doing so, we will also prove the worthiness of the categorical approach: defining external semantics in the individual-token case is considerably harder, but instead of having to do so explicitly, we will be relying on abstract results from higher category theory.

Definition 8 (Internal bound semantics – individual-token philosophy). *Let N be a Petri net, and consider $\mathfrak{F}(N)$, its corresponding FSSMC. The internal bound semantics of N in the individual-token philosophy is given by the FSSMC $\mathfrak{F}_B(N)$ generated by the same information of Definition 5.*

Even if Definition 5 and 8 look nearly identical, things are instantly complicated by the presence of symmetries. Indeed, proving comonadicity without resorting to abstract methods becomes very hard, as all naturality squares require carefully – and consistently – selecting symmetries to commute.

Proposition 3. *The assignment $\mathfrak{F}_B(-) : \mathfrak{F}(N) \mapsto \mathfrak{F}_B(N)$ defines a comonad in the category of FCMSCs and strict monoidal functors between them, \mathbf{FSSMC} .*

Proof. As we will prove in Theorem 3, the category $\mathfrak{F}_B(N)$ fits in a pullback

$$\begin{array}{ccc} \mathfrak{F}_B(N) & \xrightarrow{Q} & \mathbf{Span} \\ \varepsilon_N \downarrow & \lrcorner & \downarrow U \\ \mathfrak{F}(N) & \xrightarrow{N^\sharp} & \mathbf{Span} \end{array}$$

We will use the universal property of such pullback to find a comultiplication and a counit map and to show the comonad laws; an evident candidate to be a counit is the arrow $\varepsilon_N : \mathfrak{F}_B(N) \rightarrow \mathfrak{F}(N)$; a candidate comultiplication map $\alpha_N : \mathfrak{F}_B(N) \Rightarrow \mathfrak{F}_B^2(N)$ is obtained from the diagram

$$\begin{array}{ccc}
 \mathfrak{F}_B(N) & \xrightarrow{Q} & \mathbf{Span.} \\
 \searrow \alpha & & \downarrow U \\
 \mathfrak{F}_B^2(N) & \xrightarrow{Q_1} & \mathbf{Span.} \\
 \varepsilon_1 \downarrow \lrcorner & & \downarrow U \\
 \mathfrak{F}_B(N) & \xrightarrow{N^\sharp_1} & \mathbf{Span}
 \end{array}$$

As the unique red arrow filling the diagram above, if N^\sharp_1 is the common value of the diagonal $\varepsilon_N \circ N^\sharp = Q \circ U$ as in Section 4, and the pair (Q_1, ε_1) is obtained iterating the pullback of Section 4.

Now that we have the candidate maps (obviously natural in their component, because they are defined via a universal property that is functorial in $\mathfrak{F}(N)$), we recall the comonad laws in the component N :

$$\begin{array}{ccc}
 \mathfrak{F}_B(N) & \xrightarrow{\alpha_N} & \mathfrak{F}_B^2(N) \\
 \alpha_N \downarrow & & \downarrow \alpha_{\mathfrak{F}_B(N)} \\
 \mathfrak{F}_B^2(N) & \xrightarrow{\mathfrak{F}_B(\alpha_N)} & \mathfrak{F}_B^3(N)
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathfrak{F}_B^2(N) & \xrightarrow{\mathfrak{F}_B(\varepsilon_N)} & \mathfrak{F}_B(N) \\
 \varepsilon_{\mathfrak{F}_B(N)} \downarrow & \lrcorner & \downarrow \alpha_N \\
 \mathfrak{F}_B(N) & \xrightarrow{\alpha_N} & \mathfrak{F}_B^2(N)
 \end{array}$$

The laws are shown as follows:

- The coassociativity requires us to compute the arrows $\mathfrak{F}_B(\alpha_N)$ and $\alpha_{\mathfrak{F}_B(N)}$: First, we build the definition of $\mathfrak{F}_B^3(N)$, as the upper left corner in the diagram

$$\begin{array}{ccc}
 \mathfrak{F}_B^3(N) & \xrightarrow{Q_2} & \mathbf{Span.} \\
 \alpha_{\mathfrak{F}_B(N)} \uparrow & \downarrow \varepsilon_2 & \parallel \\
 \mathfrak{F}_B^2(N) & \xrightarrow{Q_1} & \mathbf{Span.} \\
 \alpha_N \uparrow & \downarrow \varepsilon_1 & \parallel \\
 \mathfrak{F}_B(N) & \xrightarrow{Q} & \mathbf{Span.} \\
 \varepsilon_N \downarrow & & \downarrow U \\
 \mathfrak{F}(N) & \xrightarrow{N^\sharp} & \mathbf{Span}
 \end{array}$$

Now, according to the definitions, $\alpha_{\mathfrak{F}_B(N)}$ is the arrow working as a right inverse of ε_2 , whereas

$\tilde{\mathfrak{F}}_B(\alpha_N)$ results from the diagram

$$\begin{array}{ccccc}
 \tilde{\mathfrak{F}}_B(N) & \xrightarrow{\quad} & \tilde{\mathfrak{F}}_B^3(N) & \xrightarrow{Q_2} & \mathbf{Span}_\bullet \\
 & \searrow \alpha_N & \nearrow \tilde{\mathfrak{F}}_B(\alpha_N) & \downarrow \varepsilon_2 & \parallel \\
 & & \tilde{\mathfrak{F}}_B^2(N) & \xrightarrow{Q_1} & \mathbf{Span}_\bullet \\
 & & \downarrow \varepsilon_1 & & \downarrow U \\
 & & \tilde{\mathfrak{F}}_B^2(N) & \xrightarrow{N^\sharp_2} & \mathbf{Span} \\
 & & \nearrow \alpha_N & & \parallel \\
 \tilde{\mathfrak{F}}_B(N) & \xrightarrow{N^\sharp_1} & \mathbf{Span} & & \mathbf{Span}
 \end{array}$$

The coassociativity property is implied from the fact that

$$\begin{cases} \alpha_N \circ \alpha_{\tilde{\mathfrak{F}}_B(N)} \circ \varepsilon_2 = \alpha_N \circ \tilde{\mathfrak{F}}_B(\alpha_N) \circ \varepsilon_2 \\ \alpha_N \circ \alpha_{\tilde{\mathfrak{F}}_B(N)} \circ Q_2 = \alpha_N \circ \tilde{\mathfrak{F}}_B(\alpha_N) \circ Q_2 \end{cases} \quad (1)$$

since the projection maps (ε_2, Q) from a pullback are jointly monic (as always when computing a limit). Eq. (1) hold because of the commutativities found so far:

$$\begin{aligned}
 \alpha_N \circ \alpha_{\tilde{\mathfrak{F}}_B(N)} \circ \varepsilon_2 &= \alpha_N \circ \text{id}_{\tilde{\mathfrak{F}}_B^2(N)} \\
 &= \text{id}_{\tilde{\mathfrak{F}}_B(N)} \circ \alpha_N \\
 &= \alpha_N \circ \varepsilon_1 \circ \alpha_N \\
 &= \alpha_N \circ \tilde{\mathfrak{F}}_B(\alpha_N) \circ \varepsilon_2 \\
 \alpha_N \circ \alpha_{\tilde{\mathfrak{F}}_B(N)} \circ Q_2 &= \alpha_N \circ \alpha_{\tilde{\mathfrak{F}}_B(N)} \circ Q_2 \circ \text{id}_{\mathbf{Span}_\bullet} \\
 &= \alpha_N \circ \alpha_{\tilde{\mathfrak{F}}_B(N)} \circ \varepsilon_2 \circ Q_1 \\
 &= \alpha_N \circ \text{id}_{\tilde{\mathfrak{F}}_B^2(N)} \circ Q_1 \\
 &= \alpha_N \circ Q_1 \circ \text{id}_{\mathbf{Span}_\bullet} \\
 &= \alpha_N \circ \tilde{\mathfrak{F}}_B(\alpha_N) \circ Q_2
 \end{aligned}$$

- The left and right counit laws follow from a similar chain of reasoning, and all in all from the definition of the counit and comultiplication. \square

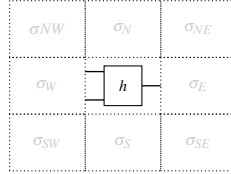
As for the commutative case, we now want to provide a semantics that is the external counterpart of Definition 8. Definition 6 can be ported almost verbatim:

Definition 9 (Non-local semantics – individual-token philosophy). *Let N be a Petri net and let \mathcal{S} be a monoidal bicategory. A Petri net with a non-local semantics is a couple (N, N^\sharp) , with N^\sharp a lax-monoidal-lax functor $\tilde{\mathfrak{F}}(N) \rightarrow \mathcal{S}$.*

A morphism $(N, N^\sharp) \rightarrow (M, M^\sharp)$ of Petri nets with non-local semantics is a strict monoidal functor $\tilde{\mathfrak{F}}(N) \xrightarrow{F} \tilde{\mathfrak{F}}(M)$ making the obvious triangle commute. Overloading notation, we denote the category of Petri nets with a non-local semantics with $\mathbf{Petri}^{\mathcal{S}}$.

Yet, actually producing a lax-monoidal-lax functor $\mathfrak{F}(N) \rightarrow \mathbf{Span}$ akin to the one in Definition 7 is not as easy: As pointed out in Example 2, in the commutative case, it doesn't really matter which object we apply a given morphism to, but only how many objects of some sort are turned into objects of some other sort. This allowed us to represent morphisms between anti-places using spans with the preimage of χ in their tips, and kept things manageable at the level of intuition.

When non-trivial symmetries are around, the causal relationship between morphisms becomes meaningful: for instance, $f \circledast g$ and $f \otimes g$ will, in general, act very differently on objects. Moreover, looking at the fibre of χ over a morphism generator f , we can describe it quite explicitly as the set of all (possibly empty) stackings of string diagrams



where various symmetries $\sigma_W, \sigma_E, \sigma_{NE}, \dots$ “surround” f on each side. If f is an identity, say over an object A , then $\chi^{-1}(1_A)$ coincides with the set of all symmetries of $\mathfrak{F}(N)$, making it clear that a lax-monoidal-lax semantics for the individual-token case won't preserve identities strictly.

4.1 External semantics via abstract machinery

With such a great deal of technicality, generalizing Definition 5 “manually” can easily go wrong. As we did already in [9], we solve the problem by inverting our reasoning: We will use abstract machinery to obtain a lax-monoidal-lax functor from the counit $\mathfrak{F}_B(N) \rightarrow \mathfrak{F}(N)$ of the comonad in Proposition 3, in such a way that the isomorphism with $\mathfrak{F}_B(N)$ will be guaranteed after internalizing.

Definition 10 (Total category, [15]). *Let \mathcal{C} be a 1-category, regarded as a locally discrete bicategory, and let $F : \mathcal{C} \rightarrow \mathbf{Span}$ be a lax monoidal-lax functor. The total category of F is the 1-category $\int F$ defined as the pullback (in \mathbf{Cat})*

$$\begin{array}{ccc} \int F & \longrightarrow & \mathbf{Span}_\bullet \\ \varepsilon_F \downarrow & \lrcorner & \downarrow U \\ \mathcal{C} & \xrightarrow{F} & \mathbf{Span} \end{array}$$

where \mathbf{Span}_\bullet is the bicategory of spans between pointed sets, and U is the forgetful functor. More concretely, $\int F$ is defined as the category² having

- 0-cells of $\int F$ are couples (X, x) where X is a 0-cell of \mathcal{C} and $x \in FX$.
- 1-cells $(X, x) \rightarrow (Y, y)$ of $\int F$ are couples (f, s) where $f : X \rightarrow Y$ is a 1-cell of \mathcal{C} and $s \in S_{Ff}$ with $Ff(s) = (x, y)$. Representing a span as a function $(S, s) \rightarrow (X \times Y, (x, y))$ between (pointed) sets, a morphism is a pair (f, s) such that $Ff : s \mapsto (x, y)$.

Definition 11. *Let \mathcal{C} and \mathcal{D} be 1-categories, and let $F : \mathcal{D} \rightarrow \mathcal{C}$ be a functor. There is a lax functor $\Gamma F : \mathcal{C} \rightarrow \mathbf{Span}$, with \mathcal{C} regarded as a locally discrete bicategory, defined as follows:*

- On objects, $C \in \mathcal{C}$ is mapped to the set $\{D \in \mathcal{D} \mid FD = C\}$.
- On morphisms, $C \xrightarrow{f} C'$ in \mathcal{C} is mapped to the span:

$$\{D \in \mathcal{D} \mid FD = C\} \xleftarrow{\text{dom}} \{g \in \mathcal{D} \mid Fg = f\} \xrightarrow{\text{cod}} \{D \in \mathcal{D} \mid FD = C'\}$$

²All 2-cells are identities, due to the 2-discreteness of \mathcal{C} .

The following fact was first observed by J. Bénabou in [5] (see also [12, Th. 5.4.5]) where instead of **Span** there is the bicategory of profunctors:

Theorem 2 ([15]). *For any category \mathcal{C} , There is an equivalence of categories*

$$\Gamma : \mathbf{Cat}/\mathcal{C} \simeq \mathbf{Cat}_l[\mathcal{C}, \mathbf{Span}] : \int$$

where the right hand side is the 1-category of lax monoidal functors and functional (i.e. whose components are all functions) natural transformations, and \int, Γ are defined as in Definition 10 and Definition 11, respectively. In particular, for $F : \mathcal{D} \rightarrow \mathcal{C}$, $\int \Gamma F$ is isomorphic to \mathcal{D} .

Relying upon Definition 11, we are ready to define our external semantics.

Definition 12 (External bound semantics – individual-token philosophy). *We define a lax-monoidal-lax functor $N^\sharp : \mathfrak{F}(N) \rightarrow \mathbf{Span}$ as $\Gamma \varepsilon_N$, Where $\varepsilon_N : \mathfrak{F}_B(N) \rightarrow \mathfrak{F}(N)$ is the functor sending any anti-place to the monoidal unit, and generating morphisms to themselves.*

Proposition 4. *The functor of Definition 12 is lax-monoidal-lax.*

Proof. That N^\sharp is lax is true by definition. We need to prove that it laxly preserves the monoidal structure. This is readily done by noticing that N^\sharp sends all objects to the set of multisets on the set of object generators of $\mathfrak{F}_B(N)$ which are in the form A^- . This set is isomorphic to S^\otimes , the set of objects of $\mathfrak{F}(N)$, and we obtain an obvious laxator $S^\otimes \times S^\otimes \rightarrow S^\otimes$ by concatenating strings.

The laxator diagram commutes only up to a 2-cell, which is to be expected given that we are working with lax functors. This 2-cell is readily obtained from the following inclusion

$$\{h \in \mathcal{D} \mid Fh = f \otimes g\} \subseteq \{h \in \mathcal{D} \mid Fh = f\} \times \{h \in \mathcal{D} \mid Fh = g\}$$

Between the span tips of $N^\sharp(f \otimes g)$ and $N^\sharp f \times N^\sharp g$, respectively. □

Theorem 3. *Let (N, N^\sharp) be an object of $\mathbf{Petri}_B^{\mathbf{Span}}$. The category $\mathfrak{F}_B(N)$ of Definition 5 is isomorphic to the category resulting from the following pullback in **Cat**:*

$$\begin{array}{ccc} \int N^\sharp & \longrightarrow & \mathbf{Span}_\bullet \\ \varepsilon_{N^\sharp} \downarrow & \lrcorner & \downarrow U \\ \mathfrak{F}(N) & \xrightarrow{N^\sharp} & \mathbf{Span} \end{array}$$

where \mathbf{Span}_\bullet is the bicategory of spans between pointed sets, and U is the forgetful functor. Explicitly:

- Objects of $\int N^\sharp$ are couples (X, x) where X is a object of \mathcal{C} and $x \in N^\sharp X$.
- Morphisms $(X, x) \rightarrow (Y, y)$ of $\int N^\sharp$ are couples (f, s) with $f : X \rightarrow Y$ a morphism of \mathcal{C} and $s \in S_{N^\sharp, f}$ such that $N^\sharp f(s) = (x, y)$.

Proof. Unrolling definitions, this is just Theorem 2. □

5 Conclusion and future work

In this work, we proved how the functorial semantics approach developed in [10] to model guarded nets can be generalized to include other traditionally familiar constructions such as bounding. To achieve such a result, we refined a technique already developed in [9] by endowing nets with a *lax-monoidal-lax* functorial semantics. This improvement is a conceptually relevant leap: if in [10] strong monoidal functors endowed tokens with properties that could be only locally true, in [9] and here tokens can be endowed with properties about the global state of the net (such as distributions of tokens in other places). The laxity requirement, which translates into the possibility to consider ensembles of tokens as a single thing, builds a tension between the intrinsic local nature of tokens and their global properties.

Mathematically, our main contribution consists in proving how the formalism of functorial semantics is mature enough to scale to complicated situations, such as non-local semantics, in the presence of symmetries; this requires adopting a higher-categorical point of view, and resorting to the entirely abstract description of many constructions that we were able to implement naively in the collective-token case.

As for directions for future work, we want to investigate what else our formalism can cover, hierarchic nets [11] and nets with inhibitor arcs [24, 23] being the most promising candidates. In general, we are now convinced of the following fact, which we consider a significant conceptual contribution:

Studying the categorical semantics of extensions of Petri nets amounts to classifying lax functors
 $\mathcal{C} \rightarrow \mathbf{Span}$ with \mathcal{C} free (commutative, symmetric) monoidal.

References

- [1] J. C. Baez, F. Genovese, J. Master, and M. Shulman. *Categories of Nets*. 2021. arXiv: [2101.04238](#) [cs, math] (cit. on p. 4).
- [2] J. C. Baez and J. Master. “Open Petri Nets”. In: *Mathematical Structures in Computer Science* 30.3 (2020), pp. 314–341. arXiv: [1808.05415](#) (cit. on p. 4).
- [3] J. C. Baez and B. S. Pollard. “A Compositional Framework for Reaction Networks”. In: *Reviews in Mathematical Physics* 29.09 (2017), p. 1750028. arXiv: [1704.02051](#) (cit. on p. 1).
- [4] P. Baldan, F. Bonchi, and F. Gadducci. “Encoding Asynchronous Interactions Using Open Petri Nets”. In: *CONCUR 2009 - Concurrency Theory*. Ed. by M. Bravetti and G. Zavattaro. Vol. 5710. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009, pp. 99–114 (cit. on p. 4).
- [5] J. Bénabou. “Introduction to Bicategories”. In: J. Bénabou, R. Davis, A. Dold, J. Isbell, S. MacLane, U. Oberst, and J. -. Roos. *Reports of the Midwest Category Seminar*. Vol. 47. Berlin, Heidelberg: Springer Berlin Heidelberg, 1967, pp. 1–77 (cit. on p. 11).
- [6] R. Bruni, H. Melgratti, U. Montanari, and P. Sobocinski. “Connector Algebras for C/E and P/T Nets’ Interactions”. In: *Logical Methods in Computer Science* 9.3 (2013). Ed. by J.-P. Katoen. arXiv: [1307.0204](#) (cit. on p. 4).
- [7] F. Genovese, A. Gryzlov, J. Herold, M. Perone, E. Post, and A. Videla. *Computational Petri Nets: Adjunctions Considered Harmful*. 2019. arXiv: [1904.12974](#) [cs, math] (cit. on p. 3).
- [8] F. Genovese and J. Herold. “Executions in (Semi-)Integer Petri Nets Are Compact Closed Categories”. In: *Electronic Proceedings in Theoretical Computer Science* 287 (2019), pp. 127–144 (cit. on p. 4).
- [9] F. Genovese, F. Loregian, and D. Palombi. *Nets with Mana: A Framework for Chemical Reaction Modelling*. 2021. arXiv: [2101.06234](#) [cs, math, q-bio] (cit. on pp. 2, 4, 10, 12).

- [10] F. Genovese and D. I. Spivak. “A Categorical Semantics for Guarded Petri Nets”. In: *Graph Transformation*. Ed. by F. Gadducci and T. Kehrer. Vol. 12150. Lecture Notes in Computer Science. Cham: Springer International Publishing, 2020, pp. 57–74 (cit. on pp. 2, 4, 5, 12).
- [11] K. Jensen and L. M. Kristensen. *Coloured Petri Nets*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009 (cit. on pp. 1, 12).
- [12] F. Loregian. “Coend Calculus”. In: *to appear in LMS Lecture Notes in Mathematics* (2020). arXiv: [1501.02503 \[math\]](https://arxiv.org/abs/1501.02503) (cit. on p. 11).
- [13] J. Master. “Petri Nets Based on Lawvere Theories”. In: *Mathematical Structures in Computer Science* 30.7 (2020), pp. 833–864. arXiv: [1904.09091](https://arxiv.org/abs/1904.09091) (cit. on p. 3).
- [14] J. Meseguer and U. Montanari. “Petri Nets Are Monoids”. In: *Information and Computation* 88.2 (1990), pp. 105–155 (cit. on pp. 2, 4).
- [15] D. Pavlović and S. Abramsky. “Specifying Interaction Categories”. In: *Category Theory and Computer Science*. Ed. by E. Moggi and G. Rosolini. Red. by G. Goos, J. Hartmanis, and J. van Leeuwen. Vol. 1290. Lecture Notes in Computer Science. Berlin, Heidelberg: Springer Berlin Heidelberg, 1997, pp. 147–158 (cit. on pp. 10, 11).
- [16] C. Petri and W. Reisig. *Petri Net*. Scholarpedia. 2008. URL: http://www.scholarpedia.org/article/Petri_net (cit. on p. 1).
- [17] W. Reisig. *Understanding Petri Nets: Modeling Techniques, Analysis Methods, Case Studies*. Berlin, Heidelberg: Springer-Verlag, 2013. 230 pp. (cit. on pp. 1, 2).
- [18] R.-C. Riemann. *Modelling of Concurrent Systems: Structural and Semantical Methods in the High Level Petri Net Calculus*. Munchen: Herbert Utz Verlag, 1999. 252 pp. (cit. on p. 1).
- [19] V. Sassone. “On the Category of Petri Net Computations”. In: *TAPSOFT '95: Theory and Practice of Software Development*. Ed. by P. D. Mosses, M. Nielsen, and M. I. Schwartzbach. Red. by G. Goos, J. Hartmanis, and J. Leeuwen. Vol. 915. Berlin, Heidelberg: Springer Berlin Heidelberg, 1995, pp. 334–348 (cit. on p. 4).
- [20] C. J. Schommer-Pries. *The Classification of Two-Dimensional Extended Topological Field Theories*. 2014. arXiv: [1112.1000 \[math\]](https://arxiv.org/abs/1112.1000) (cit. on p. 5).
- [21] Statebox Team. *The Mathematical Specification of the Statebox Language*. 2019. arXiv: [1906.07629 \[cs, math\]](https://arxiv.org/abs/1906.07629) (cit. on p. 1).
- [22] R. J. van Glabbeek. “Petri Nets, Configuration Structures and Higher Dimensional Automata”. In: *CONCUR'99 Concurrency Theory*. Ed. by J. C. M. Baeten and S. Mauw. Red. by G. Goos, J. Hartmanis, and J. van Leeuwen. Vol. 1664. Lecture Notes in Computer Science. Berlin, Heidelberg: Springer Berlin Heidelberg, 1999, pp. 21–27 (cit. on p. 2).
- [23] D. A. Zaitsev. “Universal Petri Net”. In: *Cybernetics and Systems Analysis* 48.4 (2012), pp. 498–511 (cit. on p. 12).
- [24] D. A. Zaitsev. *A Small Universal Petri net*. 2013. arXiv: [1309.1274](https://arxiv.org/abs/1309.1274) (cit. on p. 12).