A Categorical Semantics of Fuzzy Concepts in Conceptual Spaces

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We define a symmetric monoidal category modelling fuzzy concepts and fuzzy conceptual reasoning within Gärdenfors' framework of conceptual (convex) spaces. We propose log-concave functions as models of fuzzy concepts, showing that these are the most general choice satisfying a criterion due to Gärdenfors and which are well-behaved compositionally. We then generalise these to define the category of log-concave probabilistic channels between convex spaces, which allows one to model fuzzy reasoning with noisy inputs, and provides a novel example of a Markov category.

1 Introduction

How can we model conceptual reasoning in a way which is formal and yet reflects the fluidity of concept use in human cognition? One answer to this question is given by Peter Gärdenfors' framework of *conceptual spaces* [8, 9, 10], in which domains of conceptual reasoning are modelled by mathematical spaces and concepts are described geometrically, typically as *convex* regions of these spaces.

The theory of conceptual spaces is defined only semi-formally, giving room for many authors to define their own mathematical formalisations [1, 18, 23, 15, 2]. A notable aspect of the framework is that it is *compositional* in the sense that each overall conceptual space is given by composing various simpler *domains* (e.g. colour, sound, taste). This aspect makes the framework highly suited to formalisation in terms of *monoidal categories*.

Bolt et al. [3] have presented a categorical model of conceptual spaces within the *DisCoCat* framework for natural language semantics [5], using the compact monoidal category **ConvRel** of convex relations. Here a conceptual space is modelled as a convex algebra A and the meaning of a word (concept) as a convex subset. The Bolt et al. model demonstrates the use of monoidal categories in modelling the composition of conceptual spaces, and the correlations between domains contained within concepts.

However, like most formalisations of conceptual spaces, the model of [3] is limited to describing only what we may call *crisp* concepts, which are such that any point of the conceptual space either strictly is or is not a member, with no 'grey areas'. In contrast, most discussions of concepts in the cognitive science literature acknowledge that concepts should be *fuzzy* or *graded* in the sense that for any point x the degree of membership of a concept C should form a scalar value $C(x) \in [0,1]$. For example, Gärdenfors suggests defining fuzzy membership based on distance from a central region [10] representing a *prototype* [19].

In this work we propose a mathematical definition of fuzzy concepts which is compositionally well-behaved and contains crisp concepts (convex regions) as a special case. Specifically we propose that fuzzy concepts on a space X should be given by (measurable) *log-concave* functions $C\colon X \to [0,1]$. We prove that these are essentially the smallest class of functions which are closed compositionally and which satisfy the criterion of *quasi-concavity*, identified implicitly by Gärdenfors, which ensures that any point z lying 'in-between' two points x, y belongs to the concept 'as much' as they do.

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Beyond concepts, a categorical approach is well-suited to describing *processes* between spaces. To describe fuzzy processes mathematically, one typically works in a the symmetric monoidal category **Prob** whose objects are measurable spaces and morphisms are probabilistic *channels* $f: X \to Y$ [14, 11, 16]. These send each point x of X to a (sub-)probability measure (distribution) over Y. In this work, to model fuzzy *conceptual* processes we introduce *log-concave channels*, and prove that they form a symmetric monoidal subcategory **LCon** of **Prob**. In particular, the *effects* on a space X in **LCon** correspond precisely to fuzzy concepts in our sense, while the *states* of X correspond to the widely studied class of *log-concave probability measures* over the space X [20, 13]. The latter include many standard distributions such as Gaussians, allowing us to model 'noisy inputs' to our processes. More general morphisms $X \to Y$ in **LCon** may be seen as transformations of fuzzy concepts.

There are many avenues for further exploration of **LCon** as a model of fuzzy conceptual processes, such as in the modelling of metaphors as maps between conceptual spaces, and in describing concepts formed by neural network systems with noisy inputs such as β-VAEs[12]. More broadly, **LCon** may be of wider use in categorical probability theory as a novel example of a *Markov category* [7].

Related work Our work extends the model of Bolt et al. [3] to fuzzy concepts. Other such extensions include [22], which considers arbitrary measurable functions into the interval $[-1,1]$, and [4], which works with 'generalised relations', rather than measure theory. Our definition of fuzzy concept is inspired by that of Bechberger and Kuhnberger [2], though they replace convexity by star-shapedness.

Structure of article We recall convex conceptual spaces and crisp concepts (Section 2), before proposing and justifying our definition of fuzzy concepts as log-concave functions (Section 3). Next we recap categorical probability theory (Section 4), before defining the category **LCon** of log-concave channels as a model of conceptual processes (Section 5). Our main results Theorems 14 and 15 prove that **LCon** is the 'largest' monoidal category whose effects are fuzzy concepts. We close by constructing examples of log-concave channels (Section 6) and giving a toy example of conceptual reasoning (Section 7).

2 Conceptual Spaces

Peter Gärdenfors' framework of *conceptual spaces* provides an approach to the modelling of human and artificial conceptual reasoning, motivated by the cognitive sciences, and mathematically based on the notion of 'convexity' [9, 10]. In this approach, a conceptual space C (such as that of images, foods or people) is described as a product of typically simpler spaces called 'domains' (such as those of colours, sounds, tastes, temperatures ...). Based on psychological experiments, and arguments around learnability, *concepts* are modelled as regions of a conceptual space which are *convex*, meaning that any point lying in-between two instances of a concept is also an instance of that concept. In this article we work with an abstract definition of a conceptual space, without explicit reference to domains.

We begin from the formalisation due to Bolt et al. in terms of 'convex algebras' [3]. Formally, these are algebras for the finite distribution monad. In detail, for any set X we write $D(X)$ for the set of formal finite convex sums $\sum_{i=1}^{n} p_i|x_i\rangle$ of elements x_i of X, where each $p_i \in [0,1]$ with $\sum_{i=1}^{n} p_i = 1$. These formal sums satisfy natural conditions suggested by the notation: for example, the order of the $p_i|x_i\rangle$ is irrelevant, and the sum is equal to $|x_i\rangle$ when $p_i = 1$.

Definition 1. A *convex algebra* is a set X coming with a function $\alpha: D(X) \to X$ satisfying

$$\alpha(|x\rangle) = x \qquad \alpha(\sum_i p_i \alpha(\sum_j q_{i,j}|x_{i,j}\rangle)) = \alpha(\sum_{i,j} p_i q_{i,j}|x_{i,j}\rangle)$$

For any elements $x_i \in X$ and positive weights p_i with $\sum_i p_i = 1$ we may thus define a convex combination

$$\sum_{i=1}^{n} p_i x_i := \alpha(\sum_{i=1}^{n} p_i |x_i\rangle) \in X \tag{1}$$

We will denote binary convex combinations by

$$x +_p y := px + (1-p)y$$

for $x, y \in X$ and $p \in [0,1]$. A map of convex algebras $f \colon X \to Y$ is called *affine* when $f(\sum_{i=1}^{n} p_i x_i) = \sum_{i=1}^{n} p_i f(x_i)$ for all convex combinations.

To discuss fuzzy notions later, we will require the tools of probability theory, and thus consider spaces which are *measurable*. Recall that a *measurable space* is a set X with a σ-algebra $\Sigma_X \subseteq \mathbb{P}(X)$, a family of subsets, which are called *measurable*, which contains X itself and is closed under complements and countable unions. A map of measurable spaces $f \colon X \to Y$ is *measurable* if $f^{-1}(M) \in \Sigma_X$ whenever $M \in \Sigma_Y$. Our basic model of a conceptual space is now the following.

Definition 2. By a *convex space* we mean a convex algebra (X, α) which is also a measurable space. A *crisp concept* of X is a measurable subset C which is *convex*, meaning that whenever $x_1, \dots, x_n \in C$ then $\sum_{i=1}^{n} p_i x_i \in C$ also. We denote the set of crisp concepts of X by $\mathsf{Con}(X)$.

Lemma 3. Let X be a convex algebra. Then the convex subsets of X themselves forms a convex algebra via the Minkowski sum:

$$A +_p B := \{a +_p b \mid a \in A, b \in B\} \tag{2}$$

for $p \in [0,1]$. Hence if X is a convex space such that $A +_p B$ is measurable for all crisp concepts A, B, then $\mathsf{Con}(X)$ forms a convex algebra.

Examples 4. Let us consider some examples of convex spaces and their concepts; for more see [3].

1. The unit interval $[0,1]$ forms a convex space with concepts as sub-intervals.
2. Any normed vector space $(X, \|-\|)$ forms a convex space via its Borel σ-algebra, which is generated by the open subsets. In particular $X = \mathbb{R}^n$ forms a convex space with either its Borel or Lebesgue σ-algebras. The concepts are (measurable) convex subsets in the usual sense.
3. Any convex measurable subset (crisp concept) C of a convex space X is again a convex space.
4. Any convex algebra (X, α) forms a convex space by using the *discrete* σ-algebra $\Sigma_X = \mathbb{P}(X)$.
5. Any join semi-lattice $(X, \vee)$ forms a convex algebra (and hence space) by taking $x +_p y = x \vee y$ for all $p \in (0,1)$ [3], with concepts as $\vee$-closed subsets. This allows one to consider discrete convex spaces, such as truth values $\{0, 01\}$.
6. The *product* of convex spaces X, Y is the convex space on $X \times Y$ with operations

$$\sum_{i=1}^{n} p_i(x_i, y_i) = (\sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i)$$

for $x_i \in X$, $y_i \in Y$, and equipped with the *product* σ-algebra $\Sigma_{X \times Y}$, the algebra generated by the subsets of the form $A \times B$ for $A \in \Sigma_X$ and $B \in \Sigma_Y$. In particular when $C \in \mathsf{Con}(X), D \in \mathsf{Con}(Y)$ then $C \times D \in \mathsf{Con}(X \times Y)$.

7. In [3] toy conceptual spaces of colours and tastes are defined as follows. *Colour space* is defined as the 3-dimensional cube

$$C = [0,1]^3 = \{(R,G,B) \mid 0 \leq R,G,B \leq 1\}$$

of red-green-blue intensities. Specific points include (pure) green $g := (0,1,0)$, yellow $:= (1,1,0)$ etc. We can define a crisp concept 'green' for example as the (convex) open ball $G = B^{\varepsilon}_{g}$ around green of a given radius $\varepsilon > 0$, or more sharply as the singleton $\{g\}$. A simple *taste space* T is defined as the convex space (simplex) in $\mathbb{R}^4$ generated by the four points sweet, bitter, salt, and sour

$$T = \{(t_1,t_2,t_3,t_4) \mid t_i \geq 0, \sum t_i = 1\}$$

By taking the product of these convex spaces, we can form a toy *food space* as

$$F = C \times T$$

in which each food is modelled by a concept relating its colours and tastes.

8. Any set of *exemplar* points E in a convex space define a convex set via their *convex closure* $\overline{E}$, which is defined as the intersection of all convex subsets $C \subseteq X$ containing E, or equivalently as its set of convex combinations

$$\overline{E} = \{\sum_{i=1}^{n} p_i e_i \mid e_i \in E\}$$

In spaces such as $\mathbb{R}^n$, the set $\overline{E}$ will be a closed crisp concept. We can think of $\overline{E}$ as a concept 'learned' from these exemplars, with the convex closure allowing us to infer new instances of the concept.

3 Fuzzy Concepts

The concepts described so far have been crisp, or 'sharp', in that every element $x \in X$ either is or is not a member of the concept, with either $x \in C$ or $x \notin C$. Real-life concept membership is arguably a more 'fuzzy' notion, taking a value in the range $[0,1]$. Thus a concept should instead be a 'fuzzy set', a map

$$C\colon X \to [0,1]$$

where $C(x) \in [0,1]$ denotes the extent to which x is an instance of the concept C. Such mappings are partially ordered, point-wise with $C \leq D$ whenever $C(x) \leq D(x) \forall x$.

Now fuzzy concepts should not be arbitrary mappings, but respect the convex structure of X appropriately. Gärdenfors has suggested one structural feature that fuzzy concepts should satisfy, which amounts to the following requirement[1] [9].

Criterion 5. [9] Let X be a convex space. Fuzzy concepts $C\colon X \to [0,1]$ should be *quasi-concave*, meaning that for all $x, y \in X$, $p \in [0,1]$ we have

$$C(x +_p y) \geq \min\{C(x), C(y)\}$$

Equivalently, each *t-cut* $C_t := \{x \in X \mid C(x) \geq t\}$ is a convex subset of X, for $t \in [0,1]$.

[1] The analogous criterion for fuzzy *star-shaped* sets is considered in [2]

This requirement is a natural one, stating that if x and y are both members of a concept to degree $t \in [0,1]$, then so is any point lying 'between' them. Practically, it allows one to understand a fuzzy concepts C in terms of its 'cuts' $\{C_t\}_{t\in[0,1]}$, ensuring that these will indeed form crisp concepts.

However, quasi-concavity is not fully sufficient if we wish to develop a compositional theory of fuzzy concepts, due to the following observation. For any $[0,1]$-valued maps C,D on X,Y we define $C \otimes D$ on $X \times Y$ by $(x,y) \mapsto C(x)D(y)$.

Remark 6 (Quasi-concavity is not compositional)**.** For quasi-concave functions C on X and D on Y, the map $C \otimes D$ is generally not quasi-concave. For example, take $X = Y = [0,1]$ with $C(x) = 1 - \frac{x}{2}$ and $D(y) = \frac{y^2+1}{2}$. Then $C \otimes D$ acts as $(0,0),(1,1) \mapsto \frac{1}{2} > 0.46875 \leftarrow\!\shortmid (\frac{1}{2},\frac{1}{2})$.

Hence we require a stricter definition to ensure that concepts may be composed. Luckily, there is a well-known class of quasi-concave functions which provide a compositionally well-behaved definition of fuzzy concept.

Definition 7 (Log-Concavity/Fuzzy Concepts)**.** Let X be a convex algebra. A function $f\colon X \to \mathbb{R}$ is *log-concave* when for all $x,y \in X$ and $p \in [0,1]$ we have

$$f(x +_p y) \geq f(x)^p f(y)^{1-p}.$$

We define a *fuzzy concept* on a convex space X to be a measurable log-concave function $C\colon X \to [0,1]$. We denote the set of fuzzy concepts on X by $\mathsf{FCon}(X)$.

Any function f which is *concave*, with $f(x +_p y) \geq f(x) +_p f(y)$, is log-concave. Any log-concave function is quasi-concave. A function f is log-concave iff

$$\log \circ f\colon X \to [-\infty,\infty]$$

is concave, or equivalently if $f(x) = e^{u(x)}$ with $u(x)$ concave. Log-concave functions on spaces $\mathbb{R}^n$ form are well-studied class in statistics, including many standard functions from probability theory [20, 13]. They are known to be well-behaved under compositional operations such as products, convolutions and marginalisation.

Our definition of fuzzy concept is justified by the following result. Write $\mathsf{QuasCon}(X)$ for the set of quasi-concave functions $X \to [0,1]$.

Theorem 8 (Log-Concavity is canonical)**.** The sets $\mathbf{C}(X) = \mathsf{FCon}(X)$ are the largest choice of a set $\mathbf{C}(X)$ of measurable functions $X \to [0,1]$ on each convex space X, which together satisfy the following.

1. Each $\mathbf{C}(X) \subseteq \mathsf{QuasCon}(X)$;
2. $C \otimes D \in \mathbf{C}(X \otimes Y)$ whenever $C \in \mathbf{C}(X)$, $D \in \mathbf{C}(Y)$;
3. $\mathbf{C}([0,1])$ contains all affine functions $[0,1] \to [0,1]$ and/or all exponentials $x \mapsto \lambda^{-x}$ for $\lambda \geq 1$.

Proof. Appendix A. □

Hence if we accept the criterion of quasi-concavity, wish fuzzy concepts to be closed under tensors, and include a few basic examples, then log-concave functions are the broadest definition we can take.

Examples 9. Let us meet some examples of fuzzy concepts.

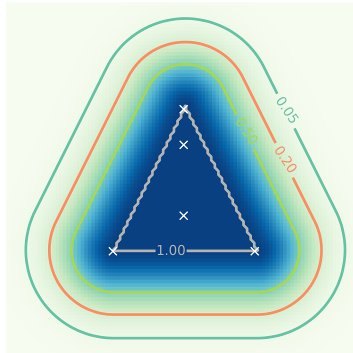


Figure 1:
Visualization of a fuzzy concept in $\mathbb{R}^2$. From a set of exemplars (white crosses) we form their convex closure, yielding the crisp concept given by the inner triangle. We then form a Gaussian fuzzification as in (3). Each t-cut of the concept (within the contour lines) is a (convex) crisp concept.

1. Crisp concepts $M \subseteq X$ correspond precisely to fuzzy concepts 1_M on X taking values only in $\{0,1\}$, via

$$1_M(x) = \begin{cases} 1 & x \in M \\ 0 & x \notin M \end{cases}$$

   Indeed one may see that 1_M is log-concave iff M is convex, and measurable iff M is. We will often identify a crisp concept $M \subseteq X$ with its fuzzy concept $1_M \colon X \to [0,1]$.

2. Any affine measurable map $X \to [0,1]$ is concave and hence a fuzzy concept.

3. Let X be a normed space with metric d, and $P \subseteq X$ a closed crisp concept. Then for any $\sigma^2 \geq 0$ we can define a Gaussian 'fuzzification' of P as the fuzzy concept

$$N^P_\sigma(x) := e^{-\frac{1}{2\sigma^2} d_H(x,P)^2} \tag{3}$$

   where d_H denotes the Hausdorff distance $d_H(x,A) := \inf_{a \in A} d(x,a)$ for $A \subseteq X$. Here P provides the 'prototypical' region in which the concept takes values 1. The concept tends to 0 as we move away from P at a rate determined by the variance σ^2. The limit case $\sigma = 0$ corresponds to the crisp concept 1_P. An example plot of such a fuzzy concept is shown in Figure 1.

   Many statistical functions besides Gaussians are log-concave also, providing alternative 'fuzzification' procedures to (3).

4. For any Hilbert space $\mathscr{H}$, any quantum effect $a \in B(\mathscr{H})$ with $0 \leq a \leq 1$ provides an (affine) fuzzy concept $\mathsf{Tr}(a-)$ on the convex space of density matrices of $\mathscr{H}$.

5. Let L be a finite semi-lattice viewed as a convex space with $\Sigma_L = \mathbb{P}(L)$. A fuzzy concept on L is a monotone map $L \to [0,1]$.

4 Probabilistic Channels

Our next goal is to introduce a category of fuzzy *processes* between spaces. To do so, in this section we must briefly recall the categorical treatment of fuzzy (probabilistic) mappings, also known as 'channels'.

Recall that a finite *measure* on a measurable space (X, Σ_X) is a function $\omega \colon \Sigma_X \to \mathbb{R}$ which is additive on countable disjoint unions, with $\omega(\emptyset) = 0$. A *subprobability measure* has $\omega(X) \leq 1$ while a *probability measure* has $\omega(X) = 1$. These provide a general notion of 'distribution' over such a space X. A standard approach to probability is to work in the following category, known as that of *probabilistic relations* [16], or more abstractly as the Kleisli category of the (sub-)*Giry Monad* [11, 14].

Definition 10. In the symmetric monoidal category category **Prob** the objects are measurable spaces (X, Σ_X) and morphisms $f \colon X \to Y$, are *channels*, also known as *Markov kernels*. These are functions $f \colon X \times \Sigma_Y \to [0,1]$ such that

1. $f(x, -) \colon \Sigma_Y \to [0,1]$ is a subprobability measure on Y, for each $x \in X$;
2. $f(-, M) \colon X \to [0,1]$ is measurable, for each $M \in \Sigma_Y$.

We write f for both the morphism and kernel, and at times write $f(x) := f(x, -)$. Thus a channel sends each $x \in X$ to a '(sub)probability distribution' $f(x)$ over Y, in a measurable way. Given another channel $g \colon Y \to Z$, the composite channel $(g \circ f) \colon X \to Z$ is defined by

$$(g \circ f)(x, M) := \int_{y \in Y} g(y, M) d f(x)(y)$$

for each $x \in X, M \in \Sigma_Z$. The identity channel $X \to X$ sends each $x \in X$ to the *point measure*

$$\delta_x(M) = \begin{cases} 1 & x \in M \\ 0 & x \notin M \end{cases}$$

The unit object I is the singleton set, with $X \otimes Y = X \times Y$. For channels $f \colon X \to W$ and $g \colon Y \to Z$ we define $f \otimes g \colon X \otimes Y \to W \otimes Z$ by

$$(f \otimes g)((x, y), (A, B)) = f(x, A) g(y, B) \tag{4}$$

for each $x \in X, y \in Y$, $A \in \Sigma_W, B \in \Sigma_Z$. Since the measures $f(x), g(y)$ are finite, this in fact specifies $(f \otimes g)(x, y)$ over $\Sigma_{W \times Z}$ uniquely as the *product measure* $f(x) \otimes g(y)$ of the measures $f(x)$ and $g(y)$.

As special cases, *states* $\omega \colon I \to X$ of X may be identified with a sub-probability measures over X, *effects* $C \colon X \to I$ with measurable functions $C \colon X \to [0,1]$, and *scalars* $I \to I$ with probabilities $p \in [0,1]$.

5 The Category of Log-Concave Channels

We can now generalise our notion of fuzzy concept to define a symmetric monoidal category of 'fuzzy conceptual processes'.

Definition 11 (Log-Concave Channels)**.** We call a channel $f \colon X \to Y$ between convex spaces X, Y *log-concave* when its kernel $f \colon X \times \Sigma_Y \to [0,1]$ is log-concave on convex subsets. That is, we have

$$f(x +_p y, A +_p B) \geq f(x, A)^p f(y, B)^{1-p} \tag{5}$$

for all $x, y \in X$ and convex $A, B \in \Sigma_Y$ for which $A +_p B$ is measurable.

We also call such a channel a *conceptual channel*. Note that a conceptual channel $C \colon X \to I$ is precisely a fuzzy concept on X. Many more examples are given in the next section.

Definition 12 (The Category **LCon**)**.** We define **LCon** to be the symmetric monoidal subcategory of **Prob** whose objects are convex spaces and whose morphisms are log-concave channels.

To establish that **LCon** is indeed a well-defined category is non-trivial, being based on an extension of a central result in the study of log-concave functions, the Prékopa-Leindler inequality.

Theorem 13 (Extended Prékopa-Leindler Inequality)**.** Let X be a convex space, $p \in (0,1)$, and $\mu, \nu\, \omega$ be σ-finite measures on X satisfying

$$\mu(A) \geq \nu(B)^p \omega(C)^{1-p} \tag{6}$$

whenever $A,B,C \in \Sigma_X$ with $A \supseteq B +_p C$ measurable. Next let $f,g,h\colon X \to \mathbb{R}_{\geq 0}$ be measurable functions satisfying

$$f(x +_p y) \geq g(x)^p h(y)^{1-p} \tag{7}$$

for all $x, y \in X$. Then

$$\left(\int_X f\, d\mu\right) \geq \left(\int_X g\, d\nu\right)^p \left(\int_X h\, d\omega\right)^{1-p} \tag{8}$$

Further, if f,g,h are quasi-concave functions then the same conclusion holds assuming only that (6) holds for $A,B,C \in \Sigma_X$ which are all convex.

Proof. Appendix B. □

The usual form of the Prékopa-Leindler inequality is the special case of the above result where $\mu = \nu = \omega$ is given by the Lebesgue measure on $X = \mathbb{R}^n$, and is proved by induction on n. In appendix B we given an alternative proof, generalising the result to the multiple measures μ, ν, ω and beyond $\mathbb{R}^n$. Using this we can now establish our main result.

Theorem 14. LCon is a well-defined symmetric monoidal subcategory of **Prob**.

Proof. Appendix B. □

We can also extend Theorem 8 to show that our definition of conceptual channels is not arbitrary, but that **LCon** is 'the largest' subcategory of **Prob** whose effects can form fuzzy concepts. Let us call a convex space *well-behaved* if for all convex measurable subsets A, B, the convex set

$$\{(a +_p b, p) \mid a \in A, b \in B, p \in [0,1]\} \subseteq X \times [0,1] \tag{9}$$

is measurable. We conjecture that every normed space, with its Borel σ-algebra, is well-behaved.

Theorem 15 (Log-Concave Channels are Canonical)**.** Let $\mathbf{C}$ be a symmetric monoidal subcategory of **Prob** containing only well-behaved convex spaces, as well as the space $[0,1]$, and for each object X write $\mathbf{C}(X) := \mathbf{C}(X,I)$. Suppose that either of the following hold.

1. $\mathsf{Con}(X) \subseteq \mathbf{C}(X) \subseteq \mathsf{FCon}(X)$ for all X;
2. $\mathsf{Con}(X) \subseteq \mathbf{C}(X) \subseteq \mathsf{QuasCon}(X)$ for all X, and $\mathbf{C}([0,1])$ contains all affine (or exponential) functions.

Then there are symmetric monoidal inclusions $\mathbf{C} \hookrightarrow \mathbf{LCon} \hookrightarrow \mathbf{Prob}$.

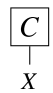
Proof. Appendix B. □

Remark 16. The proof of Theorem 15 works by showing that log-concave channels satisfy an analogue of 'complete positivity'. If a channel f is such that each channel $f \otimes \mathrm{id}_X$ preserves fuzzy concepts under post-composition, for all objects X (or even just $X = [0,1]$), then f must be log-concave.

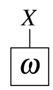
6 Examples of Log-Concave Channels

To make sense of the definition of log-concave channel and illustrate working in the category **LCon** we now give numerous examples of its morphisms. We use the *graphical calculus* for symmetric monoidal categories, in which morphisms $A \to B$ are boxes with lower input wire A and upper output wire B (read bottom to top), with I corresponding to the empty diagram [21].

1. **Effects.** Scalars $I \to I$ are values $p \in [0,1]$, and fuzzy concepts on X correspond precisely to effects



2. **States.** A state

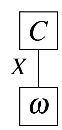


is a sub-probability measure ω on X satisfying

$$\omega(A +_p B) \geq \omega(A)^p \omega(B)^{1-p} \tag{10}$$

for all convex $A, B \in \Sigma_X$ for which $A +_p B \in \Sigma_X$. Measures for which this holds for *arbitrary* $A, B \in \Sigma_X$ are called *log-concave measures*, and are well-studied with log-concave functions [13]. Thus states on X are essentially log-concave sub-probability measures.

Given a fuzzy concept C on X, the scalar



$$= \int_{x \in X} C(x) d\omega(x) \in [0,1]$$

is the extent to which the concept C is deemed to hold over the 'distribution of inputs' ω.

3. **States from densities.** When X is (a convex measurable subset of) $\mathbb{R}^n$, it is well-known that log-concave measures ω correspond precisely to log-concave *densities*, as follows. If $\rho \colon X \to \mathbb{R}_{\geq 0}$ is a measurable log-concave function we may define a log-concave measure $\omega := \rho \lambda^X$ on X by

$$\omega(A) = \int_A \rho d\lambda^X \tag{11}$$

for each $A \in \Sigma_X$, where λ^X is the Lebesgue measure on X. Conversely, every log-concave measure on X is of the form $\omega = i \circ \omega'$ where Y is a measurable convex subset, $\omega' = \rho \lambda^Y$ for a log-concave density ρ on Y, and i is the inclusion $Y \hookrightarrow X$. Specifically Y is the affine closure of ω's *support*.

Many standard probability distributions on $\mathbb{R}^n$ form states in **LCon**, including:

(a) Each point measure δ_x;

(b) Each uniform distribution over a compact convex compact subset C, with density $1_C(x)$ and measure $\omega(A) = \frac{\lambda(A \cap C)}{\lambda(C)}$ where λ is the Lebesgue measure;

(c) Each multivariate Gaussian distribution, which (on its affine support) has log-concave density

$$\rho(x) = \frac{1}{\kappa} e^{-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)} \tag{12}$$

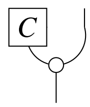
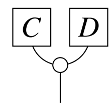
with mean $\mu \in X = \mathbb{R}^n$, covariance matrix Σ and normalisation $\kappa = \sqrt{((2\pi)^n \det(\Sigma)};$

(d) The logistic, extreme value, Laplace and chi distributions on $\mathbb{R}$, all with log-concave densities.

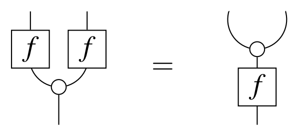
4. **Markov category maps.** Each convex space X comes with log-concave *copying* and *discarding* channels which form a commutative comonoid

and are defined by $x \mapsto \delta_{(x,x)}$ and $x \mapsto 1$ for all x, respectively. This makes **LCon** a *Markov category* in the sense of [7][2]. The presence of discarding tells us that *marginals* of log-concave channels are again log-concave, which is well-known for log-concave measures.

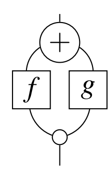
5. **Conceptual updates.** Copying lets us turn any fuzzy concept C into an 'update by C' map, (left-hand below) as well as point-wise multiply any pair of fuzzy concepts C, D.

 $:: x \mapsto C(x)\delta_x$  $:: x \mapsto C(x)D(x)$ (13)

6. **Deterministic maps.** Any *partial affine map* $f\colon X \to Y$, meaning a crisp concept $\mathsf{dom}(f) \subseteq X$ and measurable affine map $f\colon \mathsf{dom}(f) \to Y$, induces a log-concave channel $\hat{f}\colon X \to Y$ with $\hat{f}(x) = \delta_{f(x)}$ whenever $f(x)$ is defined, and $\hat{f}(x) = 0$ otherwise. In Lemma 17 in the appendix we show these are precisely the log-concave channels f which are *crisp* in that each $f(x)$ is either zero or a point measure. These crisp maps f are *deterministic* in the sense of Markov categories, satisfying

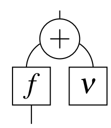


7. **Convolutions.** For any pair of log-concave channels $f, g\colon X \to Y$ between vector spaces we may define their *convolution* $f \star g$ as the log-concave channel

 (14)

where $+$ is the monoid $(x,y) \mapsto x + y$. When interpreting f and g as sending each $x \in X$ to a random variable over Y, $f \star g$ sends each element to the sum of these random variables.

8. **Noisy maps.** As a case of the previous example, given any (measurable) partial affine map $f\colon X \to Y$, now viewed as a channel, and any state ν of Y, we can form a log-concave channel

 (15)

which sends $(x, A) \mapsto \nu(A - f(x))$. If ν models 'random noise' over the space Y, then this channel describes a random variable $y = fx + \nu$ in terms of input $x \in X$. Considering spaces $\mathbb{R}^n$ and maps (15) where f is linear and ν is a Gaussian (noise) probability measure yields the symmetric monoidal category **Gauss** of Gaussian probability theory from [6]. Thus **Gauss** $\hookrightarrow$ **LCon**.

[2] More precisely, the subcategory of maps which preserve discarding form a Markov category.

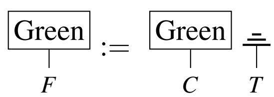
9. **Channels from densities.** Let X,Y be convex measurable subsets of $\mathbb{R}^n, \mathbb{R}^m$ respectively, and $\rho : X \times Y \to \mathbb{R}_{\geq 0}$ be a measurable log-concave function such that $\int_Y \rho(x,y)dy \leq 1$ for each $x \in X$, where dy denotes the Lebesgue measure on Y. Then we may define a log-concave channel by

$$f(x,A) = \int_A \rho(x,y)dy \tag{16}$$

for each $A \in \Sigma_Y$. This follows from the usual the Prékopa-Leindler inequality in $\mathbb{R}^n$ (Lemma 19 in the appendix). It would be interesting to find a converse to this result, analogous to that for states.

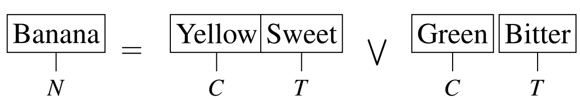
7 Toy Application: Reasoning in Food Space

In closing we demonstrate a toy example of conceptual reasoning in **LCon**, returning to our example of 'food space' $F = C \otimes T$ from Example 4 (7), based on [3]. As in that example, let us first define a crisp concept Green $= B_g^\varepsilon$ of radius $\varepsilon = 0.1$ in C around pure green $g = (0,1,0)$. We can extend this to a crisp concept on the whole of F via



In the same way we define crisp concepts 'Yellow', 'Sweet' and 'Bitter' on F.

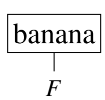
Now suppose an agent wishes to learn the concept of 'banana' from a set of exemplars in F containing a banana they conceptualise as yellow and sweet, as well as another they deem to be green and bitter. They form a crisp concept B by taking the convex closure of these concepts



(17)

where $C \vee D = \overline{C \cup D}$ is the convex closure, the join in the partial inclusion order on crisp concepts of F.

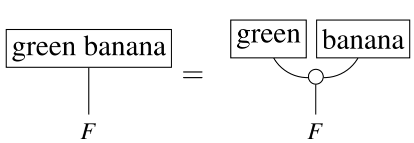
Fuzzifying concepts Since they are uncertain about the definition of their new concept 'banana', the agent may wish to replace their concept with a fuzzy one. They can convert all of their crisp concepts into fuzzy ones using the 'Gaussian fuzzification' of Example 9 (3). For example, we define a fuzzification 'banana' of the crisp concept 'Banana' with variance σ_{Ba}^2 as



$$:= N_{\sigma_{Ba}}^{\text{Banana}} :: (c,t) \mapsto e^{-\frac{1}{2\sigma^2} d_H((c,t),\text{Banana})^2}$$

We define fuzzy concepts 'green', 'yellow', 'bitter' and 'sweet' via σ_G, $\sigma_Y, \sigma_{Bi}, \sigma_S, \sigma_{Ba}$ similarly.

Combining fuzzy concepts We can combine any of our fuzzy concepts using the copying maps, as in (13). For example, we can define a fuzzy concept 'green banana' as



$$:: x \mapsto \text{green}(x)\text{banana}(x)$$

In Figure 2 we plot some examples of composite fuzzy concepts on the food space F.

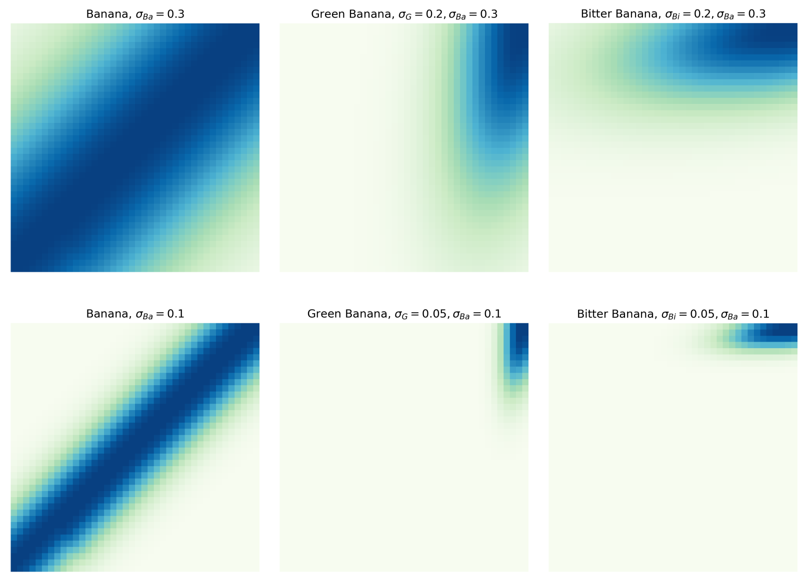
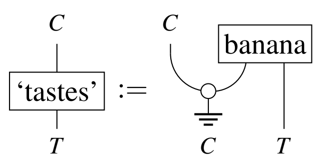
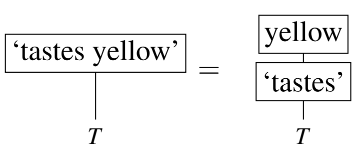


Figure 2:
Fuzzy concepts in food space, for differing variance parameters, plotted over the unit square $[\text{yellow}, \text{green}] \times [\text{sweet}, \text{bitter}] \subseteq F$. Decreasing variance increase the crispness of the concepts. Values range from dark blue (1) through green to white (0).

A taste-colour channel As a first example of conceptual reasoning beyond simply combining concepts, we consider an example of a 'metaphorical' mapping between domains. Consider the channel from tastes to colours defined by



where $\stackrel{-}{\equiv}_C$ is the uniform (Lebesgue) measure over C. This channel transforms any concept on colours into one tastes via precomposition. For example we can interpret the concept of 'tasting yellow' as



$$:: t \mapsto \frac{1}{\lambda(C)} \int_{c \in C} \text{yellow}(c)\text{banana}(c,t)d\lambda(c)$$

where λ denotes the Lebesgue measure on C. In future work it would be interesting to explore more sophisticated examples of conceptual (log-concave) channels between conceptual (convex) spaces, including those with a linguistic interpretation as metaphors. It would also be desirable to extend the learning process (17) to give a 'join' on fuzzy concepts, rather than merely crisp ones.

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A Proofs

Proof of Lemma 3. If $x = a +_p b$ and $y = a' +_p b'$ for $a, a' \in A, b, b' \in B$ one may see straightforwardly that, for any $r \in [0,1]$, we have $x +_r y = \hat{a} +_{(rp+(1-r)q)} \hat{b}$ for some $\hat{a} \in A$ and $\hat{b} \in B$, using convexity of A and B. □

Proof of Theorem 8. We have already seen that log-concave functions satisfy all of these properties. Conversely, suppose we have such a chosen set of functions $\mathbf{C}(X)$ on each convex space X. Fix a convex space X and let $C \in \mathbf{C}(X)$, with $x, y \in X$ and $p \in [0,1]$. We will show that

$$C(x +_p y) \geq C(x)^p C(y)^{1-p} \tag{18}$$

so that C is log-concave. If either $C(x)$ or $C(y)$ is zero this is trivial, otherwise without loss of generality suppose $c := \frac{C(y)}{C(x)} < 1$. Suppose that $h\colon [0,1] \to [0,1]$ is a function satisfying

$$\begin{cases} h(0) = \lambda \\ h(p) = \lambda c^p \\ h(1) = \lambda c \end{cases} \tag{19}$$

for some $\lambda \in (0,1)$, and such that $C \otimes h$ is quasi-concave. Then by rescaling, for simplicity we may assume $\lambda = 1$. Then we have

$$\begin{aligned} C(x +_p x')h(p) &\geq \min\{C(x)h(0), C(x')h(1)\} \\ &= \min\{C(x), C(x')c\} = C(x) \end{aligned}$$

Multiplying both sides by $\frac{C(x')}{C(x)}^{1-p}$ then yields precisely (18).

Now suppose that $\mathbf{C}([0,1])$ contains the exponential functions as above. Then $h(x) = c^x$ for $x \in [0,1]$ satisfies (19) and by assumption $C \otimes h \in \mathbf{C}(X \otimes [0,1])$, making it quasi-concave, and so we are done.

Next suppose instead that $\mathbf{C}([0,1])$ contains the affine functions. To establish the result, we will show for any $p, c \in (0,1)$ that there exists a quadratic polynomial h with real roots satisfying (19), with $h([0,1]) \subseteq [0,1]$. Then by rescaling further if necessary this means that we can write $h(x) = q(x)r(x)$ where $q(x) = ax + b$, $r(y) = cy + d$ are affine functions $[0,1] \to [0,1]$.

By assumption the function $C \otimes q \otimes r\colon (x,y,z) \mapsto C(x)q(y)r(z)$ belongs to $\mathbf{C}(X \otimes [0,1] \otimes [0,1])$, making it quasi-concave. Since the map $y \mapsto (y,y)$ is affine, this means that $C \otimes h\colon X \otimes [0,1] \to [0,1]$ is quasi-concave also, and we are done as before.

Finally we must verify that there exists such a quadratic polynomial h. Let $d := \frac{c^p - c - (1-c)(1-p)}{p(1-p)}$ and define

$$g(x) = (1 - c + dx)(1 - x) + c = -dx^2 + (c + d - 1)x + 1$$

Then $g(0) = 1$, $g(1) = c$ and $g(p) = c^p$. Note that d is positive; after rearranging the numerator this is equivalent to noting that

$$c^p - c \geq 1 - c - p + cp$$

which holds since

$$p(c-1) \leq c - 1 \leq c^p - 1$$

Hence g has negative quadratic term, and so has two real roots, and since $g(x)$ is strictly positive for $x = 0,1$ these roots must lie outside of $[0,1]$. So we can write $g(x) = k(x-a)(b-x)$ where $a < 0, b > 1$

and $k > 0$. Hence we have $g(x) = Kq(x)q'(x)$ where q and q' are affine functions $[0,1] \to [0,1]$, for appropriate positive scalar K. Thus $h(x) = q(x)q'(x)$ is our desired function in $\mathbf{C}([0,1])$, and we are done. $\square$

Write **PConv** for the category of convex spaces and partial affine maps $X \to Y$, with $\mathsf{dom}(g \circ f) := f^{-1}(\mathsf{dom}(g))$, and $\mathbf{LCon}_{\mathsf{Cr}}$ for the subcategory of crisp channels in **LCon**.

Lemma 17. There is an equivalence of categories $\mathbf{PConv} \simeq \mathbf{LCon}_{\mathsf{Cr}}$.

Proof. Let $f\colon X \to Y$ be crisp in **Prob**, and define a partial function $g\colon X \rightharpoonup Y$ with $f(x) = \delta_g(x)$ whenever $f(x)$ is defined. Then $\mathsf{dom}(g) = f(-,X)^{-1}(1)$ is measurable, and for each $M \in \Sigma_Y$, $f(-,M)^{-1}(1) = \mathsf{dom}(g) \wedge g^{-1}(M)$ is measurable and so g is a measurable function on a measurable domain. Now log-concavity is seen to be equivalent to the requirement that for all $x,y \in \mathsf{dom}(g)$ that

$$g(x) \in A, g(y) \in B \implies g(x +_p y) \in A +_p B \tag{20}$$

Taking $A = \{g(x)\}, B = \{g(y)\}$ shows that $x +_p y \in \mathsf{dom}(g)$ with $g(x +_p y) = g(x) +_p g(y)$. Hence g is a partial affine map with $f = \hat{g}$. Conversely, if $f = \hat{g}$ for such a map then (20) is certainly satisfied and measurability of g on its domain ensures that $\hat{g}$ is a Markov kernel.

$\square$

B LCon is well-defined and canonical

Let us work towards our proof that **LCon** is a well-defined monoidal subcategory of **Prob**. We begin by recalling some basic results in measure theory. Let (X, Σ_X) be a measurable space and μ a measure on it. We say that μ is *σ-finite* if X can be written as a countable union of sets X_i with $\mu(X_i) \leq \infty$.

Proposition 18. Let (X, Σ_X, μ) be a σ-finite measure space and $f\colon X \to \mathbb{R}_{\geq 0}$ integrable. Then

$$\int f d\mu = \int_0^\infty \mu(f^{-1}(u,\infty))du = \int_{-\infty}^\infty \mu(f^{-1}(e^v,\infty))e^v dv$$

where du, dv are each the Lebesgue measure on $\mathbb{R}$.

Proof. The first equation is a well-known consequence of Fubini's theorem, applied to the space $X \times \mathbb{R}$. For the second equation, we apply integral substitution with $u = e^v$. $\square$

Next we recall the standard Prékopa-Leindler inequality on spaces $\mathbb{R}^n$.

Lemma 19 (Prékopa-Leindler inequality, [17])**.** Let $0 < p < 1$ and f,g,h be non-negative bounded measurable functions on $\mathbb{R}^n$ satisfying (7) for all $x,y \in \mathbb{R}$. Then

$$\int_{\mathbb{R}^n} f(z)dz \geq \left(\int_{\mathbb{R}^n} f(x)dx\right)^p \left(\int_{\mathbb{R}^n} h(y)dy\right)^{1-p}$$

We now extend this result to the form of Theorem 13.

Proof of Theorem 13. First observe that if $t,t' \in \mathbb{R}$ with $g(x) > e^t$ and $h(y) > e^{t'}$ then

$$f(x +_p y) \geq g(x)^p h(y)^{1-p} > (e^t)^p (e^{t'})^{(1-p)} = e^{t +_p t'}$$

Hence we have

$$f^{-1}(e^{t+_p t'},\infty) \supseteq g^{-1}(e^t,\infty) +_p h^{-1}(e^{t'},\infty)$$

Now by assumption on the measures we have

$$\begin{aligned}\mu(f^{-1}(e^{t+_p t'},\infty))e^{t+_p t'} &\geq \nu(g^{-1}(e^t,\infty))^p \omega(h^{-1}(e^{t'},\infty))^{1-p} e^{t+_p t'} \\ &= (\nu(g^{-1}(e^t,\infty))\cdot e^t)^p(\omega(h^{-1}(e^{t'},\infty))e^{t'})^{1-p}\end{aligned}$$

for all $t,t' \in \mathbb{R}$. Applying Proposition 18 with the one-dimensional form of the Prékopa-Leindler inequality (Lemma 19) we have

$$\begin{aligned}\int_X f d\mu &= \int_0^\infty \mu(f^{-1}(e^v,\infty))\cdot e^v dv \\ &\geq \left(\int_{-\infty}^\infty \nu(g^{-1}(e^t,\infty))\cdot e^t dt\right)^p \left(\int_{-\infty}^\infty \omega(h^{-1}(e^{t'},\infty))\cdot e^{t'} dt'\right)^{1-p} \\ &= \left(\int_X g d\nu\right)^p \left(\int_X h d\omega\right)^{1-p}\end{aligned}$$

as required. For the final statement observe that if f,g,h are quasi-concave then each subset $r^{-1}(e^t,\infty)$ for $r = f,g,h$ will be convex. □

Remark 20. The above provides an alternative proof of the usual Prékopa-Leindler inequality assuming only its one-dimensional form and that the Lebesgue measure λ is log-concave. Typically however the Prékopa-Leindler inequality is used to establish that λ is log-concave in the first place. It would be interesting to explore whether Theorem 13 provides any novel applications of the Prékopa-Leindler inequality to more general convex spaces.

We can now establish that log-concave channels are closed under composition.

Proof of Theorem 14. All identities and coherence isomorphisms are log-concave channels since they are of the form of Lemma 17. Hence it suffices to show that for any log-concave $f\colon X \to Y$, $g\colon Y \to Z$ and any convex space W that the channel $h := (g \circ f) \otimes \mathrm{id}_W \colon X \otimes W \to Z \otimes W$ is log-concave. From the definition of **Prob** one may see that for all $(x,w) \in X \times W$ and measurable $E \subseteq Z \times W$ we have

$$h((x,w),E) = \int_{y \in Y} g(y,E^w) df(x,y)$$

where $E^w := \{z \in Z \mid (z,w) \in E\}$.

Let $x,y \in X$, $w,w' \in W$ and $p \in (0,1)$. By definition the measures $\mu = f(x+_p x')$, $\nu = f(x)$, $\omega = f(x')$ are all finite and satisfy

$$f(x+_p x',C) \geq f(x,A)^p f(x',B)^{1-p}$$

whenever $A,B,C \in \Sigma_Y$ are convex with $C \subseteq A +_p B$. Now let $D,E \subseteq Z \times W$ be convex and measurable and suppose that $F := D +_p E$ is measurable also. Then D^w, $E^{w'}$ will be convex also. Note that if $(x,w) \in D$ and $(x',w') \in E$ then $(x+_p x', w+_p w') \in F$. Hence $F^{w+_p w'} \supseteq C^w +_p D^{w'}$. Since g is log-concave, we conclude that for each $y,y' \in Y$ we have

$$\begin{aligned}g(y+_p y', C^{w+_p w'}) &\geq g(y+_p y', A^w +_p B^{w'}) \\ &\geq g(y,A^w)^p g(y',B^{w'})^{1-p}\end{aligned}$$

Noting also that $g(,-C^{w+_p w'})$, $g(-,A^w)$ and $g(-,B^w)$ are all log-concave and hence quasi-concave functions, we may apply the final statement of Theorem 13 with μ, ν, ω as above to give

$$\begin{aligned} h((x+_p x', w+_p w'), F) &= \int_{y \in Y} g(y, F^{w+_p w'}) df(x+_p x', y) \\ &\geq \left(\int_{y \in Y} g(y, D^w) df(x,y)\right)^p \left(\int_{y \in Y} g(y, E^{w'}) df(x', y)\right)^{1-p} \\ &= h((x,w), D)^p h((x', w'), E)^{1-p} \end{aligned}$$

Hence h is a log-concave channel as required. □

Finally we prove that log concave channels are 'canonical' as we claim.

Proof of Theorem 15. Since **C** is a monoidal subcategory of **Prob**, all effects in **C** are measurable and condition (2) of Theorem 8 holds. Hence by Theorem 8 we in fact have that (2) $\implies$ (1).

We now show that (1) ensures that every $f\colon X \to Y$ in **C** is log-concave. Let $A, B \in \Sigma_Y$ be convex. Then defining C to be the set (9), by assumption 1_C is an effect on $Y \times [0,1]$ in **C**. Hence the effect $D = 1_C \circ (f \otimes \mathrm{id})$ on $X \times [0,1]$ belongs to **C** also, and must be log-concave. Thus for any $x, y \in X$

$$f(x+_p y, A+_p B) = D(x, +_p y, p) \geq D(x,1)^p D(y,0)^{1-p} = f(x,A)^p f(y,B)^{1-p}$$

making f log-concave. □