Characterizing Double Categories of Relations

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May 2021

1 Introduction

Categories of relations and their duals, corelations, arise in both pure and applied category theory. In physics, relations play a role intermediate between the classical system of sets and functions and the fully quantum system of Hilbert spaces and bounded operators [HV20]. Corelations are the prop for certain Frobenius monoids [CF16]. Corelations have been characterized as a pushout in the anaylsis of the semantics of string diagrams [FZ18]. Relations have been described formally as "tabular allegories" [FS90] and also as "bicategories of relations" [CW87]. Recent interest in bicategories of relations appears to be stage-setting for interpretations of regular logic [FS19].

Double categories [GP99], [GP04] give another formalization of relations. However, double categories incorporate two types of morphisms on given objects related by cells. Sometimes these are special cells called "companions" or "conjoints" which give certain restrictions and extensions in the double category. The double category of relations incorporates both the classical structure of sets and the non-classical structure of the relational calculus. This makes sense for any regular category $\mathscr{C}$, resulting in a double category $\mathbb{R}\mathbf{el}(\mathscr{C})$, or even for a cartesian category $\mathscr{C}$ with a stable factorization system $\mathcal{F} = (\mathcal{E}, \mathcal{M})$, resulting in a double category $\mathbb{R}\mathbf{el}(\mathscr{C}; \mathcal{F})$.

The starting point is double categories of spans, which were characterized in [Ale18] as cartesian equipments satisfying extra conditions: the external unit map is full, strong tabulators exist, and Eilenberg-Moore objects exist for any copointed endomorphisms. Our question is: What are the conditions on an equipment or on a cartesian equipment ensuring that it is equivalent to a double category of the form $\mathbb{R}\mathbf{el}(\mathscr{C})$? Our work will show that what distinguishes $\mathbb{R}\mathbf{el}(\mathscr{C})$ from other cartesian equipments is essentially that so-called "tabulators" exist and are inclusions; that the unit of the tabulator adjunction is invertible; that the domain of any inclusion is a tabulator; and finally that a form of the Frobenius law holds for local products. Such double categories might be called "relational double categories."

Here are four potential applications.

1. Relational theories take their models in sets and relations [BPS17]. A characterization of double categories of relations should provide a means to tell which double categories support a sound interpretation of relational theories and provide a forum for a comparison of ordinary Lawvere theories and relational theories.

2. Descriptive logic is a main formalism for knowledge representation. However, bicategories of relations have been proposed as an alternative [Pat17]. Insofar as there is need for a comparison of such knowledge representation with other systems, double categories could provide an ambient structure. A characterization of double categories of relations could isolate the extra structure on a double category giving the formalism of knowledge representation and modeling interaction with other systems.

3. Any characterization of relations would afford a dual characterization of corelations, a topic of some recent interest. For example in [FZ18], corelations are shown to be a certain pushout, leading to characterizations of equivalence relations, partial equivalence relations, linear subspaces and others. Double categorical versions of these result might lead to forums for comparing different levels of structure.

4. Monoidal bifibrations give rise to certain equipments [Shu08]. It is of interest to see which equipments support an inverse construction. Our conjecture is that under such a correspondence double categories of relations correspond to subobject bifibrations. If this is the case it would be a starting point for extending fibrational semantics of type theories to interpretations in double categories with extra structure.

The following sections give an overview of some of our results. This is a report on ongoing work. Some material is in progress and might be adjusted before coming into final form.

2 Some Preliminary Observations

For any cartesian and cocartesian category $\mathscr{C}$ there is an oplax/lax adjunction $\mathbb{S}\mathbf{pan}(\mathscr{C}) \rightleftarrows \mathbb{C}\mathbf{ospan}(\mathscr{C})$ between double categories of spans and cospans. One the one hand, a span is taken to the cospan given by its pushout; and on the other, a cospan is taken to a pullback. Pushouts compute extensions in cospans and pullbacks compute restrictions in spans. An **equipment** is a double category with all restrictions and extensions.

A double category $\mathbb{D}$ admits tabulators if there is a right adjoint $\top\colon \mathbb{D}_1 \to \mathbb{D}_0$ to the external identity functor $y\colon \mathbb{D}_0 \to \mathbb{D}_1$ coming with the structure of $\mathbb{D}$. The tabulator replaces missing restrictions in $\mathbb{D}$. That is, a bare proarrow $p\colon A \nrightarrow B$ has nothing to restrict along; taking its tabulator, however, gives a replacement with a similar universal property. The theorem of Niefield [Nie12] is that the identity functor on cartesian $\mathbb{D}_0$ extends to a normalized oplax/lax adjunction $\mathbb{S}\mathbf{pan}(\mathscr{C}) \rightleftarrows \mathbb{D}$ if, and only if, $\mathbb{D}$ is an equipment with tabulators.

Some initial work on relations follows this outline. Take $\mathbb{D}_0$ to be a regular category. Start from the observation that the identity functor on $\mathbb{D}$ extends to an oplax/lax adjunction $\mathbb{R}\mathbf{el}(\mathscr{C}) \rightleftarrows \mathbb{C}\mathbf{orel}(\mathscr{C})$ provided that $\mathscr{C}$ is regular and coregular. On the one hand, F sends a relation $\langle d, c\rangle\colon R \to A \times B$ to the extension of the codiagonal on R in corelations along d and c. On the other hand, a corelation $[p, q]\colon A + B \to C$ is sent to the restriction of the diagonal relation on C along p and q. For $\mathbb{D}$ in the place of corelations, tabulators replace extensions. Our first result is that the identity functor on $\mathbb{D}_0$ extends to a normalized oplax/lax adjoint equivalence $F\colon \mathbb{R}\mathbf{el}(\mathbb{D}_0) \rightleftarrows \mathbb{D}\colon G$ if, and only if,

1. $\mathbb{D}$ is an equipment and y_e is an extension cell for each regular epimorphism e;
2. $\mathbb{D}$ has tabulators;
3. every proarrow $p\colon A \nrightarrow B$ is an extension of its tabulator;
4. every relation $R \to A \times B$ is a tabulator of its canonical extension.

The first two conditions allow the definition of F and G. The last two conditions ensure that they form an adjoint equivalance. A modified Beck-Chevalley condition is one ingredient in making this an adjoint equivalence of pseudo double functors. However, the result is only a starting point: for asking that $\mathbb{D}_0$ is regular is cheating. Additionally, it uses neither the cartesian axioms nor the Frobenius condition that characterizes bicategories of relations.

2.1 Inclusions and Covers

The condition on y_e is one ingredient that makes the proof of the oplax/lax adjunction between relations and corelations possible. It is precisely what makes F oplax. In $\mathbb{R}\mathbf{el}(\mathscr{C})$ for regular $\mathscr{C}$, it is certainly the case that all such arrows produce extensions y_e. Why? In a generic double category, an extension ξ of the unit on A along e on both sides results in a globular cell from the cokernel of e to the unit $\gamma\colon e^* \otimes e_! \Rightarrow y_E$ such that $\gamma\xi = y_e$. Extensions in $\mathbb{R}\mathbf{el}(\mathscr{C})$ in particular are computed by images. Therefore, e is a regular epimorphism if, and only if it computes the extension of y_A. In other words, e is a regular epimorphism if, and only if, the unique globular cell γ from the cokernel of e is an iso $e^* \otimes e_! \cong y_E$. Thus, make the definitions: A morphism $e\colon A \to E$ in an equipment $\mathbb{D}$ is a **cover** if the canonical globular cell is an iso $e^* \otimes e_! \cong y_E$. A morphism $m\colon E \to B$ is an **inclusion** if the canonical globular cell from the kernel of m is an iso $m_! \otimes m^* \cong y_E$ (Cf. [Sch15]). Further conditions on tabulators and inclusions will make covers and inclusions into the two classes of an orthogonal factorization system on $\mathbb{D}_0$.

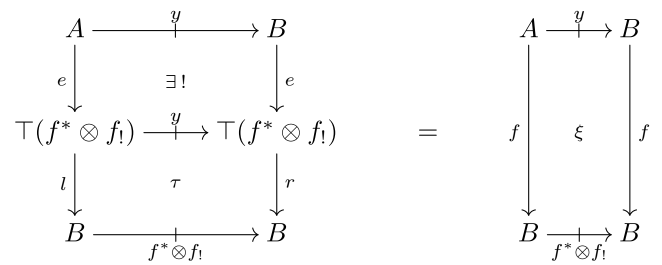
2.2 Cartesian Equipments

A double category is **cartesian** if the canonical functors $\Delta\colon \mathbb{D} \to \mathbb{D} \times \mathbb{D}$ and $\mathbb{D} \to 1$ are pseudo double functors and have right adjoints that are pseudo-double functors. A cartesian equipment is equivalent to a double category of spans if, and only if, $\mathbb{D}$ is a "unit-pure" cartesian equipment that has certain Eilenberg-Moore objects for copointed endomorphisms [Ale18]. In particular, this is the case if, and only if, the identity functor on $\mathbb{D}_0$ extends to an equivalence of double categories $\mathbb{S}\mathbf{pan}(\mathbb{D}_0) \simeq \mathbb{D}$.

The existence of Eilenberg-Moore objects implies that of tabulators. Now, our view is that tabulators govern the interaction between the two types of structure in a double category. Conditions on tabulators distinguish types of double categories. These roughly fall into two types: profunctor-type and relation-type. In the former case, every relation is its own tabulator; this is expressed by the fact that $y\colon \mathbb{D}_0 \to \mathbb{D}_1$ is fully faithful. The legs of the morphism are jointly monic; for allegories this is the condition $\phi = gf^\circ$. Finally, the tabulator should replace the missing image factorizations. Thus, owning to the centrality of this structure, our goal is to produce a characterization of cartesian equipments with tabulators that are double categories of relations.

3 Main Characterization

Require that the legs of the tabulator of any proarrow of $\mathbb{D}$ are jointly an inclusion; and that the unit of the adjunction $y \dashv \top$ is invertible. By the universal property of the tabulator there is a factorization of the cokernel of a given morphism f:



Since B presents the tabulator of y_B, it follows that $l = r$. It turns out that e is a cover and l is an inclusion. This will give an image factorization of f. Orthogonality of the two classes will follow by the fact that domains of inclusions will be required to be tabulators of their cokernels (essentially, this is the fact that domains of ordinary injective functions are isomorphic to their images).

The crucial ingredient leading to pullback-stability is a version of Frobenius reciprocity (cf. [nLa20]). For each object B of a cartesian bicategory $\mathbb{B}$ there is an induced hyperdoctrine, valued in meet-semilattices

$$\mathbb{B}(i, B)\colon \mathbf{Map}(\mathbb{B})^{op} \to \mathbf{SLat}$$

where i is the inclusion of so-called "maps" into $\mathbb{B}$. The restatement of the Frobenius Law from [CW87] in this context is the axiom

$$r \wedge qf^\circ = (rf \wedge q)f^\circ.$$

Reinterpreting this law in the context of equipments with $()^\circ = ()^*$, there is the present version of **Frobenius Reciprocity for Cartesian Equipments**:

$$r \wedge f^* \otimes q \cong f^* \otimes (f_! \otimes r \wedge q) \tag{3.1}$$

written in diagrammatic order where '$\wedge$' denotes the local product as in [Ale18]. In particular, the first main result:

Theorem 3.1. *With $\mathcal{E}$ the class of covers and $\mathcal{M}$ the class of inclusions, $\mathcal{F} = (\mathcal{E}, \mathcal{M})$ is a stable factorization system on $\mathbb{D}_0$ and thus $\mathbb{R}\mathbf{el}(\mathbb{D}_0; \mathcal{F})$ is a double category.*

Here is a proof sketch. First note that $\mathbb{D}_0$ has all pullbacks, given by the universal property of the tabulator. Pullback-stability now follows. For let $e\colon B \to C$ denote a cover and $hd = ce$ a pullback square. To see that d is a cover, calculate that

$$\begin{aligned} y &\cong y \wedge d^* \otimes c_! \otimes c^* \otimes d_! && \text{(technical lemma)} \\ &\cong d^* \otimes (d_! \wedge c_! \otimes c^* \otimes d_!) && \text{(Frobenius)} \\ &\cong d^* \otimes d_! \wedge d_! && (c_! \otimes d^* \otimes d_! \cong d_!) \\ &\cong d^* \otimes d_! && (\wedge \text{ idempotent}) \end{aligned}$$

In fact the double category $\mathbb{R}\mathbf{el}(\mathbb{D}_0; \mathcal{F})$ is a cartesian equipment satisfying Frobenius reciprocity. With this in hand, modulo some details, a rough version of the main result is:

Theorem 3.2. *Let $\mathbb{D}$ denote a cartesian equipment with tabulators. There is a stable factorization system $\mathcal{F}$ on $\mathbb{D}_0$ such that the identity functor on $\mathbb{D}_0$ extends to an adjoint oplax/lax equivalence $\mathbb{R}\mathbf{el}(\mathbb{D}_0; \mathcal{F}) \simeq \mathbb{D}$ if, and only if,*

1. *the unit $1 \Rightarrow \top y$ is fully faithful;*
2. *every tabulator is jointly an inclusion and the domain of each inclusion is the tabulator of its cokernel;*
3. *local products satisfy the Frobenius axiom.*

As with the first oplax/lax adjunction, extensions in $\mathbb{D}$ provide the oplax functor and taking tabulators provides the lax functor valued in relations. One question remains as to the precise role of Beck-Chevalley in making this a strong adjoint equivalence of pseudo-double functors. But this is left for future work.

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