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Graphical Regular Logic: the complete 2-dimensional picture

Regular logic is the fragment of first order logic generated by equality, true, conjunction, and existential quantification. In this talk we will summarise the recent revamping and completion of the project to rigorously and completely understand the connection between regular logic – manifest as the internal language of regular categories – and general *regular calculi* – those objects capturing regular theories: contexts, predicates in those contexts, and supporting the regular fragment of logic thereupon. The connection between these two notions is mediated by a graphical formalism, and by our main theorem we may understand this graphical formalism as expressing the natural rules and operations of regular logic.

In contrast to earlier work on the topic we give here a fully 2-dimensional treatment of the matter, and significantly generalise the earlier objects of study. In this way we are no longer obstructed by the technicalities imposed by working 1-dimensionally, and are able to successfully prove our desired comparison theorem, which we will state and understand in this talk: the 2-category of relational po-categories is "pseudo-reflective" in the 2-category of regular calculi. In addition, owing to the new generality we obtain the novel corollary that taking the regular category of internal functions in relational po-category is a 2-dimensionally represented 2-functor $\mathcal{R}\mathsf{lPoCat} \to \mathcal{R}\mathsf{gCat}$.

1 Introduction

Regular logic is the fragment of first order logic generated by equality (=), true (`true`), conjunction ($\wedge$), and existential quantification ($\exists$). A defining feature of this fragment is that it is expressive enough to define functions and composition of functions, or more generally composition of relations: given relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, their composite is given by the formula

$$R \fatsemi S = \{(x, z) \mid \exists y. R(x, y) \wedge S(y, z)\}.$$

Indeed, regular logic is the internal language of regular categories, which may in turn be understood as a categorical characterisation of the minimal structure needed to have a well-behaved notion of relation.

While regular categories put emphasis on the notion of binary relation, the presence of finite products allows them to handle n-ary relations—that is, subobjects of n-fold products—and their various compositions. To organise more complicated multi-way composites of relations, many fields have developed some notion of wiring diagram. A good amount of recent work, including but not limited to control theory [BE15; BSZ14; FSR16], database theory and knowledge representation [BSS18; Pat17], electrical engineering [BF18], and chemistry [BP17], all serve to demonstrate the link between these languages and categories for which the morphisms are relations.

In [FS19b], the authors took an important step in unifying the notions of wiring diagrams and regular logic. To make rigorous the idea of wiring diagrams, the authors first extended the work of [FS19c] to the context of symmetric monoidal *po-categories* – those symmetric monoidal categories enriched over poset. They then utilised the notion of "supply" for a "po-prop" to transfer the wiring language dictated by the "po-prop for wiring $\mathbb{W}$" to any symmetric monoidal po-category supplying this po-prop. Furthermore, in terms of such a supply for $\mathbb{W}$ and some additional structure, they were able to axiomatise those symmetric monoidal po-categories arising as the po-categories of relations of a regular category – the so-called "relational" po-categories. Finally the authors showed that there is a 2-dimensional equivalence of the 2-categories of regular categories and relational po-categories, thereby successfully introducing a graphical formalism for regular categories.

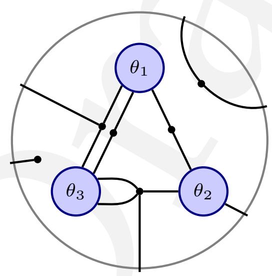
Theorem ([FS19b, Theorem 7.3])**.** *The assignment $\mathcal{R} \mapsto \mathbb{R}\mathsf{el}\,\mathcal{R}$ of a regular category to its po-category of relations is a 2-functor $\mathbb{R}\mathsf{el}\colon \mathcal{R}\mathsf{gCat} \to \mathcal{R}\mathsf{lPoCat}$. The assignment $\mathbb{R} \mapsto \mathsf{LAdj}\,\mathbb{R}$ of a relational po-category to its regular category of left adjoints forms an opposed 2-functor $\mathsf{LAdj}\colon \mathcal{R}\mathsf{lPoCat} \to \mathcal{R}\mathsf{gCat}$. Moreover, these 2-functors form an equivalence of 2-categories.*

In this paper and its companion [cFS21], our goal is to complete this work by connecting the now equivalent concepts of regular categories and relational po-categories, to the here novel notion of *regular calculi* – structures which house regular theories. A regular calculus $(\mathbb{C}_P, P)$ comprises the data of a symmetric monoidal po-category $\mathbb{C}_P$ which supplies the po-prop for wiring $\mathbb{W}$, as well as a "right ajax" po-functor P from $\mathbb{C}_P$ to posets. We think of the objects of $\Gamma \in \operatorname{Ob} \mathbb{C}_P$ as contexts for predicates in some regular theory, each poset $P(\Gamma)$ as the poset of predicates in the context Γ ordered by implication, and each morphism $f\colon \Gamma \to \Gamma'$ as a method for converting formulas in the context Γ to formulas in the context Γ' by using, among other things, equality $(=)$, true (`true`), conjunction $(\wedge)$, and existential quantification $(\exists)$.

As the symmetric monoidal po-category of contexts $\mathbb{C}_P$ of a regular calculus $(\mathbb{C}_P, P)$ supplies the po-prop for wiring $\mathbb{W}$, we have automatically a graphical language for describing predicates which arise by such manipulations. For instance, in a fixed context Γ from which we draw variables, and from predicates θ_1, θ_2, and θ_3 of arity 3, 3, and 4 respectively, we might wish to construct the formula ψ as

$$\psi(y, z, z', x, x', z'') = \exists\, \tilde{x}, \tilde{y}, \left[\theta_1(\tilde{x}, \tilde{y}, y) \wedge \theta_2(x', \tilde{x}, x) \wedge \theta_3(y, \tilde{y}, x', x') \wedge (z = z')\right] \ .$$

By using the graphical notation for regular calculi we develop here, we will be able to realise this formula as the below "graphical term" of the regular calculus.



We have already suggested that regular calculi provide a home for regular theories, and an important class of such are the regular categories – equivalently the relational po-categories. We will show that from a relational po-category $\mathbb{R}$ one may construct a regular calculus $\mathbb{P}\mathsf{rd}\,\mathbb{R}$ through a process we call *taking predicates*. Given a regular category $\mathcal{R}$, viewed as its po-category of relations $\mathbb{R}\mathsf{el}\,\mathcal{R}$, our construction yields the regular calculus $\mathbb{P}\mathsf{rd}\,\mathbb{R}\mathsf{el}\,\mathcal{R}$ whose contexts are the objects of $\mathcal{R}$ and whose predicates θ in context r are precisely the subobjects $\theta \rightarrowtail r$, exactly as we might have hoped. This assignment of relational po-categories $\mathbb{R} \mapsto \mathbb{P}\mathsf{rd}\,\mathbb{R}$ to regular calculi we extend to a 2-functor $\mathbb{P}\mathsf{rd}\colon \mathcal{R}\mathsf{lPoCat} \to \mathcal{R}\mathsf{gCalc}$.

In the other direction, given a regular calculus $(\mathbb{C}_P, P)$ we are able to construct a "syntactic po-category" $\mathbb{S}\mathsf{yn}(\mathbb{C}_P, P)$. This po-category has as objects pairs (Γ, θ) of contexts $\Gamma \in \operatorname{Ob} \mathbb{C}_P$ and predicates θ in context Γ, and so models the syntax of the regular calculus closely. This construction has many desirable properties and we are able to prove that the syntactic po-category of a regular calculus is relational, that is, it may be viewed as the po-category of relations of a regular category.

The assignment $(\mathbb{C}_P, P) \mapsto \mathbb{S}\mathsf{yn}(\mathbb{C}_P, P)$ of regular categories to their syntactic po-categories we extend to a 2-functor $\mathbb{S}\mathsf{yn}\colon \mathcal{R}\mathsf{gCalc} \to \mathcal{R}\mathsf{lPoCat}$, and with this we prove our first comparison theorem – appearing as part of Theorem 7.10 in this paper and whose proof is elaborated in the companion.

Theorem A. *The* 2*-functors* $\mathbb{S}\mathsf{yn}\colon \mathcal{R}\mathsf{gCalc} \to \mathcal{R}\mathsf{lPoCat}$ *and* $\mathbb{P}\mathsf{rd}\colon \mathcal{R}\mathsf{lPoCat} \to \mathcal{R}\mathsf{gCalc}$ *form a bi-adjunction* $\mathbb{S}\mathsf{yn} \dashv_{\mathrm{bi}} \mathbb{P}\mathsf{rd}$*.*

By a *bi-adjunction* here, we mean the appropriate notion of 2-dimensional adjunction where the equations on the unit and co-unit now hold only up to invertible 3-dimensional morphisms, each of which satisfy some appropriate equation. As adjunctions transfer a rich set of category-theoretic aspects, so too do bi-adjunctions by analogy – this bi-adjunctions affords us a rich comparison of the 2-category theory of regular calculi and relational po-categories. However, from the point of view of understanding the connection between the graphical regular logic of regular calculi, and the traditional formalism of regular categories it is as yet unsatisfactory.

Without a stronger theorem we cannot be sure that by working syntactically in the regular theory carried by the regular calculus $\mathbb{P}\mathsf{rd}\,\mathbb{R}$ we are in fact working in the relational po-category $\mathbb{R}$ itself. That is, we wish to know: is there an equivalence between the syntax $\mathbb{S}\mathsf{yn}\,\mathbb{P}\mathsf{rd}\,\mathbb{R}$ given by our graphical regular calculus approach and the relational po-category $\mathbb{R}$? To this end we prove as part of our main theorem the following general answer to this question.

Theorem B. *The co-unit of the bi-adjunction* $\mathbb{S}\mathsf{yn} \dashv_{\mathrm{bi}} \mathbb{P}\mathsf{rd}$ *is an adjoint equivalence, so relational po-categories are pseudo-reflective in regular calculi.*

By *pseudo-reflective* here we mean the appropriate 2-dimensional version of the analogous ordinary category theoretic notion of a fully-faithful inclusion of sub-categories which admits a left adjoint. Among other things, this result tells us that we may freely embed relational po-categories and their morphisms into regular calculi by taking predicates, and that all graphical manipulations and syntactical operations hold in the original object: $\mathbb{S}\mathsf{yn}\,\mathbb{P}\mathsf{rd}\,\mathbb{R} \simeq \mathbb{R}$.

As [FS19b] proves that regular categories and relational po-categories have equivalent 2-categories, we have in fact also obtained the following theorem, which appears as Corollary 7.11 in our paper.

Theorem C. *The* 2*-category of regular categories is pseudo-reflective in regular calculi.*

Finally, we prove Theorem 7.4 below which, in concert with the above theorems, allows us to prove the following, appearing as Corollary 7.5.

Theorem D. *The* 2*-functor* $\mathsf{LAdj}\colon \mathcal{R}\mathsf{lPoCat} \to \mathcal{R}\mathsf{gCat}$ *is bi-represented by the relational po-category* $\mathbb{S}\mathsf{yn}\,\mathbb{P}\mathsf{rd}\,\mathbb{W}$*.*

By *bi-represented* here we mean that there is an appropriate 2-dimensional equivalence of the 2-functors $\mathcal{R}\mathsf{lPoCat}(\mathbb{S}\mathsf{yn}\,\mathbb{P}\mathsf{rd}\,\mathbb{W}, -)$ and LAdj. This says that taking left adjoints in a relational po-category – the operation which extracts the underlying regular category – is obtained by mapping out of $\mathbb{S}\mathsf{yn}\,\mathbb{P}\mathsf{rd}\,\mathbb{W}$. This latter object is the syntax of the graphical regular logic of abstract wiring diagrams, and we see that it determines precisely the equivalence between regular categories and relational po-categories.

1.1 Outline

We have striven, where reasonable, to render this paper as self-contained as possible. Where we make use of results from the body of work of [FS19a; FS19b; FS19c] we are careful to cite them or reprove them in our context. Nevertheless, we have still chosen to defer certain proofs or developments to the companion paper [cFS21] where we felt their contribution to the narrative of this paper was relatively minor, or would be outweighed by their length or technical nature. With that in mind, this paper is organised as follows.

Section 2 presents the setting of symmetric monoidal po-categories and morphisms thereof in which we will be working. Section 3 introduces po-prop for wiring $\mathbb{W}$, the notion of supply for a po-prop, and develops our graphical notation for $\mathbb{W}$ as well as for symmetric monoidal po-categories which supply it. Then in Section 4 we define the central structures of this paper, the regular calculi and their morphisms, by way of the notions of right adjoint monoid and right ajax po-functor. In Section 5 we finally develop our graphical formalism for regular calculi by defining graphical terms and establishing key lemmas which afford us intuitive means of graphical reasoning and manipulations. Additionally in that section we sketch the construction of the syntactic po-category of a regular calculus, and discuss some of the results and obstructions proven in the companion. In Section 6 we recall the axiomatisation of relational po-categories, and construct and study the 2-functor $\mathbb{P}\mathsf{rd}$ which takes a relational po-category to its regular calculus of predicates. Finally in Section 7 we compare the 2-categories of regular calculi and relational po-categories in several ways. In this section we state the main theorem, Theorem 7.10, whose proof we defer to the companion, and using it we prove that various characterisations and corollaries of interest.

1.2 Acknowledgements

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2 Background on symmetric monoidal po-categories

To develop the theory of regular calculi and to state and prove our main results, Theorems 7.4 and 7.10, we will make extensive use of the language of symmetric monoidal po-categories and various higher morphisms thereof. In Section 2.1 below we recall briefly the needed notions of *oplax-natural transformation*, *modification*, and *adjunction in a* 2*-category*. Readers familiar with these notions are invited to omit this section. Following this, in Section 2.2 we will observe the several significant specialisations of these notions to the *po-categorical* setting and cement terminology therein. Finally, in Section 2.3 we will recall the notion of *symmetric monoidal po-category* and various morphisms thereof.

Before we proceed with this background, let us pause a moment to record the salient features of our notation in this paper. While we endeavour to be standard in most aspects, it may nevertheless be useful to note the following.

- We typically denote composition in diagrammatic order, so the composite of $f\colon A \to B$ and $g\colon B \to C$ is $f \fatsemi g\colon A \to C$. We often denote the identity morphism $\mathrm{id}_c\colon c \to c$ on an object $c \in \mathcal{C}$ simply by the name of the object, c. Thus if $f\colon c \to d$, we have $(c \fatsemi f) = f = (f \fatsemi d)$.
- We may denote the unique map from an object c to a terminal 1 as $!\colon c \to 1$, and we denote the top element of any poset P by $\mathtt{true} \in P$.
- We denote the universal map into a product by $\langle f, g\rangle$ and the universal map out of a coproduct by $[f, g]$.
- Given a natural number $n \in \mathbb{N}$, we write $\underline{n}$ for the set $\{1, 2, \ldots, n\} \in \mathsf{FinSet}$; in particular $\underline{0} = \varnothing$.
- We will write $c^{\otimes n}$ in a monoidal category to denote the left-associated n-fold iterated binary tensor product $(\cdots((c \otimes c) \otimes c)\cdots) \otimes c$.
- Given a lax monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$, we denote the *laxators* by $\varphi\colon I \to F(I)$ and $\varphi_{c,c'}\colon F(c) \otimes' F(c') \to F(c \otimes c')$ for objects $c, c' \in \operatorname{Ob}\mathcal{C}$. If F is strong, then we will make use of the same notation, but refer to these maps as *strongators* instead.
- Where our arguments make use of more than one dimension, we will write the morphisms with Latin letters, the 2-morphisms with Greek letters, and the 3-morphisms with Hebrew letters. For instance, 2-functors will be denoted by F, G, ..., oplax-natural transformations will be denoted by α, β, ..., and modifications will be denoted by $\aleph$, $\beth$, ...

2.1 Background on 2-categories

We will take for granted the notion of 2-category and 2-functor, but briefly recall here the definitions of higher morphisms between these. The reader already comfortable with such notions is nevertheless encouraged to review the various specialisations obtained in the *po-categorical* setting in Section 2.2, and the later background on *symmetric monoidal po-categories* in Section 2.3.

Definition 2.1 (Oplax-natural transformations & modifications)**.** Given a pair of parallel 2-functors $F, G\colon \mathcal{K} \to \mathcal{L}$, an **oplax-natural transformation** $\alpha\colon F \Rightarrow G$ comprises the data of object components $\alpha_c\colon Fc \to Gc$ for each object $c \in \operatorname{Ob}\mathcal{K}$, and morphism components $\alpha_h\colon (\alpha_c \fatsemi Gh) \Rightarrow (Fh \fatsemi \alpha_{c'})$ for each morphism $h\colon c \to c'$ of $\mathcal{K}$. These morphism components are required to be natural with respect to 2-morphisms of $\mathcal{K}$, and are required to be compatible with identity morphisms and composition in $\mathcal{K}$. For details see, for example, [JY20, Definition 4.2.1].

An oplax-natural transformation α is **pseudo-natural** when each morphism component α_h is a 2-isomorphism, and is 2**-natural** when each morphism component α_h is an identity.

A **modification** $\aleph\colon \alpha \Rrightarrow \beta$ between oplax-natural transformations $\alpha, \beta\colon F \Rightarrow G$ comprises the data of object components $\aleph_c\colon \alpha_c \Rightarrow \beta_c$ in $\mathcal{L}(Fc, Gc)$ for each object $c \in \operatorname{Ob}\mathcal{K}$, which are required to be compatible with the morphism components of α and β. For details see, for example, [JY20, Definition 4.4.1].

Recall that, given a 2-category $\mathcal{K}$, an **adjunction in** $\mathcal{K}$ consists of a pair of objects $c, d \in \operatorname{Ob}\mathcal{K}$, a pair of morphisms $l\colon c \to d$ and $r\colon d \to c$, and a pair of 2-morphisms $\eta\colon d \Rightarrow (l \fatsemi r)$ and $\epsilon\colon (l \fatsemi r) \Rightarrow c$ such that the following pair of diagrams, the **triangle equalities**, are rendered commutative.

$$\begin{array}{ccc} l & \xRightarrow{\eta \fatsemi l} & lrl \\ & \searrow^{=} & \Downarrow{\scriptstyle l \fatsemi \epsilon} \\ & & l \end{array} \qquad\qquad \begin{array}{ccc} r & & \\ {\scriptstyle r \fatsemi \eta}\Downarrow & \searrow^{=} & \\ rlr & \xRightarrow[\epsilon \fatsemi r]{} & r \end{array} \tag{1}$$

We are careful to avoid the $\vdash$ symbol in this context; *the symbol* $\vdash$ *always means entailment*. One may verify that adjunctions compose, and so by $\mathsf{LAdj}(\mathcal{K})$ we denote the 1-category with the same objects as $\mathcal{K}$ and whose morphisms are the data of left adjoints (l, r, η, ϵ) in $\mathcal{K}$.

For given data (l, r, η, ϵ) as above, the property of being an adjunction is expressed equationally in the compositions of the ambient 2-category. As such, we obtain the following lemma.

Lemma 2.2. *Let $F\colon \mathcal{K} \to \mathcal{L}$ be a 2-functor. The assignment $(l, r, \eta, \epsilon) \mapsto (Fl, Fr, F\eta, F\epsilon)$ sends adjunctions in $\mathcal{K}$ to adjunctions in $\mathcal{L}$, and so gives rise to a functor between the categories of left adjoints,* $\mathsf{LAdj}(F)\colon \mathsf{LAdj}\,\mathcal{K} \to \mathsf{LAdj}\,\mathcal{L}$. *Moreover, this assignment $F \mapsto \mathsf{LAdj}\,F$ of 2-functors itself 2-functorial in 2-functors and so extends to a functor* $\mathsf{LAdj}\colon 2\,\mathfrak{Cat} \to \mathfrak{Cat}$. □

In fact more is true, LAdj is a 2-functor when the 2-morphisms in $2\,\mathfrak{Cat}$ are themselves required to be left adjoints, but we will not need this fact in this generality.

2.2 Po-categories

The theory of 2-categories specialises significantly to the context of *po-categories*, and so we recall briefly the appropriate definitions now.

Definition 2.3 (Po-category)**.** A **po-category** $\mathbb{C}$ is a locally-posetal 2-category, that is, it is an ordinary 2-category $\mathbb{C}$ for which the category $\mathbb{C}(c, c')$ of morphisms between any two objects is thin: there is at most a single 2-morphism between any pair of parallel morphisms.

A **po-functor** $F\colon \mathbb{C} \to \mathbb{D}$ between po-categories is an ordinary 2-functor, but we may summarise this structure by requiring that F is an ordinary functor of the underlying 1-categories and that the functions $F_{c,c'}\colon \mathbb{C}(c, c') \to \mathbb{D}(Fc, Fc')$ are monotonic for all objects $c, c' \in \mathbb{C}$.

An **oplax-natural transformation** $\alpha\colon F \Rightarrow G$ between po-functors $F, G\colon \mathbb{C} \to \mathbb{D}$ is an ordinary oplax-natural transformation between the 2-functors F and G. However, all of the compatibility conditions are degenerate and so the data is merely a collection of morphisms $\alpha_c\colon Fc \to Gc$ which satisfy $F(f) \fatsemi \alpha_{c'} \leq \alpha_c \fatsemi G(f)$ for all morphisms $f\colon c \to c'$ of $\mathbb{C}$. In particular, a 2**-natural transformation** of po-functors is merely a natural transformation of the underlying functors.

Modifications are especially degenerate. Given parallel oplax-natural transformations $\alpha, \beta\colon F \Rightarrow G$, we write $\alpha \leq \beta$ if for each $c \in \mathbb{C}$ there is an inequality $\alpha_c \leq \beta_c$ in $\mathbb{D}(Fc, Gc)$ between c-components. Thus we are motivated in writing $[\mathbb{C}, \mathbb{D}]$ for the po-category of po-functors, oplax-natural transformations, and modifications; we call it the **po-category of po-functors from** $\mathbb{C}$ **to** $\mathbb{D}$.

Notation 2.4 (Po-categories)**.** To distinguish po-categories from 1-categories we will write the former with double-struck letters, as in $\mathbb{C}, \mathbb{D}, \ldots$, and reserve script for the latter, as in $\mathcal{C}, \mathcal{D}, \ldots$

Note that in any 2-category, any two right adjoints to a given morphism are isomorphic, so in a po-category, a given morphism has *at most one* right adjoint.

Definition 2.5 (Left adjoint oplax-natural transformation)**.** Let $\mathbb{C}$ and $\mathbb{D}$ be po-categories. A **left adjoint oplax-natural transformation** is a left adjoint in the po-category $[\mathbb{C}, \mathbb{D}]$ of Definition 2.3.

As a consequence of the posetal nature of the po-category of po-functors $[\mathbb{C}, \mathbb{D}]$ we may freely pass the condition of left adjointness between oplax-natural transformations and their components, in the following sense.

Lemma 2.6. *Let $F, G\colon \mathbb{C} \to \mathbb{D}$ be po-functors, and let $\lambda\colon F \Rightarrow G$ and $\rho\colon F \Rightarrow G$ be opposed oplax-natural transformations. Then λ is left adjoint to ρ if and only if for each $c \in \mathbb{C}$ the components $\lambda_c\colon Fc \to Gc$ are left adjoint to $\rho_c\colon Gc \to Fc$ in $\mathbb{D}$.*

Proof. The forward direction is true even for 2-categories that aren't locally posetal; the backwards direction holds since the uniqueness of 2-morphisms in a po-category implies that the triangle equalities (1) hold trivially. □

Observe that an invertible modification whose eventual codomain is a po-category is necessarily an equality. As such, between po-categories the notions of 2-dimensional and 1-dimensional equivalence coincide.

Definition 2.7. An **equivalence of po-categories** is an equivalence in the 1-category of po-categories and po-functors. We say that a po-functor $F\colon \mathbb{C} \to \mathbb{D}$ is **fully-faithful** when the morphism $F_{c,c'}\colon \mathbb{C}(c, c') \to \mathbb{D}(Fc, Fc')$ is an isomorphism of posets. Furthermore, a **splitting for essentially surjectivity of** F is a specified function sending $d \in \mathrm{Ob}\,\mathbb{D}$ to a pair $(c \in \mathrm{Ob}\,\mathbb{C}, Fc \cong d)$. We may abbreviate this situation by saying that F **is split essentially surjective** to mean an implicit, specified function as before.

We will take for granted the following extension of the classical result relating fully-faithful split essentially surjective functors and equivalences.

Lemma 2.8. *Given a po-functor $F\colon \mathbb{C} \to \mathbb{D}$, the data of an equivalence on F is equivalently the data of a splitting for essential surjectivity of F and the property of fully-faithfulness for F.* □

2.3 Symmetric monoidal po-categories

We will have a great deal of use for symmetric monoidal po-categories. These objects may be viewed as 3-categories which are "petite" in two dimensions: they are locally po-categorical, and they have only one object. However, it is conceptually simpler to think of symmetric monoidal po-categories instead as symmetric monoidal 1-categories with extra structure: hom-sets are equipped with an order, and the monoidal operation is monotonic on morphisms.

Definition 2.9 (Symmetric monoidal po-category)**.** A **symmetric monoidal structure** on a po-category $\mathbb{C}$ consists of a symmetric monoidal structure $(\otimes, I, \lambda, \rho)$ on its underlying 1-category, such that $\otimes$ is a additionally a po-functor. That is, $(f_1 \otimes g_1) \leq (f_2 \otimes g_2)$ whenever $f_1 \leq f_2$ and $g_1 \leq g_2$. Recall that this means λ and ρ are automatically 2-natural (Definition 2.3).

A **strong symmetric monoidal po-functor** $(F, \varphi)\colon (\mathbb{C}, \otimes, I) \to (\mathbb{D}, \otimes', I')$ is a po-functor $F\colon \mathbb{C} \to \mathbb{D}$ whose underlying functor is strong symmetric monoidal. Recall that this means that the **strongators** $\varphi_{c,c'}\colon Fc \otimes' Fc' \to F(c \otimes c')$ are automatically 2-natural.

We will frequently wish to apply the qualifier "monoidal" to various forms of natural transformations; by a **monoidal '*adjective*' natural transformation** we will always mean an '*adjective*' natural transformation whose components additionally obey the monoidal natural transformation conditions strictly. For example, a **monoidal left adjoint oplax-natural transformation** $\alpha\colon (F, \varphi) \Rightarrow (G, \psi)$ is a left adjoint oplax-natural transformation

$\alpha\colon F \Rightarrow G$ whose components $\alpha_c\colon Fc \to Gc$ additionally obey the monoidal natural transformation conditions strictly – for instance $\varphi_I \mathbin{⨾} \alpha_I = \psi_I$.

Notation 2.10 (Symmetry isomorphisms)**.** If $(\mathbb{C}, \otimes, I)$ is a symmetric monoidal po-category, $m, n \in \mathbb{N}$ are natural numbers, and $c\colon \underline{m} \times \underline{n} \to \mathbb{C}$ is a family of objects in $\mathbb{C}$, then there is a canonical natural isomorphism

$$\sigma\colon \bigotimes_{i \in \underline{m}} \bigotimes_{j \in \underline{n}} c(i,j) \xrightarrow{\cong} \bigotimes_{j \in \underline{n}} \bigotimes_{i \in \underline{m}} c(i,j)\,. \tag{2}$$

We will refer to these σ as the **symmetry** isomorphisms, though note that generally such isomorphisms involve associators and unitors too.

As additional background for these structures and for the coming definitions, we will assume that the reader has some familiarity with the content of [FS19a; FS19b; FS19c]. Nevertheless we will endeavour to explicitly recall specific results and details from these papers when we make use of such.

3 Supplying wires

Our work toward understanding graphical regular logic begins with the establishment of the supporting machinery which was developed in [FS19b; FS19c] and whose salient details we recall here. In order to render our graphical terms, we will need already the more primitive notion of *wiring diagram*. The somehow prototypical case of these is the generic structure supporting a basic graphical calculus, viz., the *po-prop* for wiring $\mathbb{W}$. Once we have gained some proficiency in this context, we will see how the notion of *supply* for a po-prop allows us to understand wiring diagrams in any po-category which supplies $\mathbb{W}$.

3.1 The po-prop $\mathbb{W}$ for wiring

Definition 3.1 (Po-prop)**.** A **po-prop** is a symmetric strict monoidal po-category $\mathbb{P}$ whose monoid of objects is isomorphic to $(\mathbb{N}, 0, +)$. A **po-prop functor** $F\colon \mathbb{P} \to \mathbb{Q}$ is a bijective-on-objects symmetric strict monoidal po-functor.

The first, and indeed most important example we intend to consider is the po-prop for wiring. Consider the symmetric monoidal 2-category $(\mathbb{C}\mathsf{spn}^{\mathrm{co}}, \varnothing, +)$, i.e. the 2-dual of cospans between finite sets.

Definition 3.2 ($\mathbb{W}$)**.** The **po-prop for wiring**, $\mathbb{W}$, is the local poset reflection of the full and locally full sub-2-category of $(\mathbb{C}\mathsf{spn}^{\mathrm{co}}, \varnothing, +)$ spanned by the finite ordinals $\underline{n}$.

In Proposition 3.5 we shall give a more explicit description of the hom posets of $\mathbb{W}$, after which we will exhibit the generating morphisms and relations for $\mathbb{W}$ graphically.

Note that $\mathbb{W}$ is almost the prop of equivalence relations, also known as corelations—see [CF17]—but without the "extra" law, which would equate the cospans $\underline{0} \to \underline{0} \leftarrow \underline{0}$ and $\underline{0} \to \underline{1} \leftarrow \underline{0}$.

Remark 3.3. One can motivate the definition of $\mathbb{W}$ as follows. A regular category $\mathbb{R}$ has finite products, and thus each object r is equipped with morphisms $\epsilon_r : r \to 1$ and $\delta_r : r \to r \otimes r$. The category $\mathsf{FinSet}^{\mathrm{op}}$ is the free finite product category, and—in a sense that we will soon make precise—the theory of comonoids.

In the po-category of relations $\mathbb{R}\mathsf{el}(\mathcal{R})$, morphisms coming from $\mathcal{R}$ are precisely the left adjoints. It was shown in [Her00, Theorem A.2] that the span construction freely adds right adjoints, subject to the condition that pullbacks in $\mathcal{R}$ are sent to Beck-Chevalley squares. Since all of our categories are po-categories, we are using the local posetal reflection $\mathbb{W}$ of $\mathbb{S}\mathsf{pan}(\mathsf{FinSet}^{\mathrm{op}})$.

Remark 3.4. In this paper we prefer to work in the po-categorical setting as the uniqueness of 2-morphisms simplifies many of the coherence conditions. For implementation on computers, however, quotients by equivalence relations are often a source of strife. In that case, one can use $\mathbb{C}\mathsf{spn}^{\mathrm{co}}$ in place of $\mathbb{W}$ throughout. Though $\mathbb{C}\mathsf{spn}^{\mathrm{co}}$ is not a po-category, all the results go through. Roughly the reason is that the only time $\mathbb{W}$ is used is for maps $\mathbb{W} \to \mathbb{P}\mathsf{oset}$, and since $\mathbb{P}\mathsf{oset}$ is itself locally posetal, the 2-category of (right ajax) functors $\mathbb{W} \to \mathbb{P}\mathsf{oset}$ is equivalent to that of (right ajax) po-functors $\mathbb{C}\mathsf{spn}^{\mathrm{co}} \to \mathbb{P}\mathsf{oset}$.

Proposition 3.5. *The hom-posets of $\mathbb{W}$ admit the following explicit description:*

$$\mathbb{W}(\underline{m}, \underline{n}) \cong \begin{cases} \{0 \le 1\}^{\mathrm{op}} & \text{if } m = n = 0 \\ \mathrm{ER}^{\mathrm{op}}(m+n) & \text{if } m + n \ge 1 \end{cases}$$

where $\{0 \le 1\}$ is the poset of booleans, and $\mathrm{ER}(p)$ is the poset of equivalence relations on the set $\underline{p}$, ordered by inclusion.

Proof. For any m, n, may identify $\mathbb{W}(\underline{m}, \underline{n})^{\mathrm{op}}$ with the poset reflection of $\mathbb{C}\mathsf{spn}(\underline{m}, \underline{n})$. But $\mathbb{C}\mathsf{spn}(\underline{m}, \underline{n})$ is the coslice category $\underline{m+n}/\mathsf{FinSet}$, consisting of finite sets S equipped with functions $\underline{m+n} \to S$. If $m + n = 0$, then the poset reflection is that of FinSet, namely $\{0 \le 1\}$; otherwise it may be identified with the poset of equivalence relations on $\underline{m+n}$. Indeed, every function $\underline{m+n} \to S$ factors as an epi followed by a mono, and every mono out of a nonempty set has a retraction. $\square$

The proof just given suggests two further results on the structure of $\mathbb{W}$ which we record here.

Corollary 3.6. *Every morphism $\omega : \underline{n} \to \underline{m}$ in $\mathbb{W}$ with $m+n > 0$ admits a unique representation as a jointly surjective cospan of finite sets $\underline{n} \to \underline{n}_w \leftarrow \underline{m}$.* $\square$

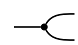
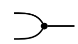
Remark 3.7. At this point we have succeeded in showing that there are in fact *canonical choices* of representative cospans for every morphism of $\mathbb{W}$. When the domain or codomain are inhabited, then the above corollary uniquely determines a jointly epimorphic cospan, and in the remaining case we choose $\underline{0} \to \underline{0} \leftarrow \underline{0}$ and $\underline{0} \to \underline{1} \leftarrow \underline{0}$ as our representatives for the two distinct elements of $\mathbb{W}(\underline{0}, \underline{0})$. Moreover, observe that between these canonical representatives there is always at most a single morphism of cospans and so we needn't concern ourselves with representatives at this level.

Corollary 3.8. *The two distinct elements of $\mathbb{W}(\underline{0}, \underline{0})$ both serve as monoidal identities for morphisms. That is,*

$$\omega + (\underline{0} \to \underline{0} \leftarrow \underline{0}) = \omega = \omega + (\underline{0} \to \underline{1} \leftarrow \underline{0}) \colon \underline{n} \to \underline{m}$$

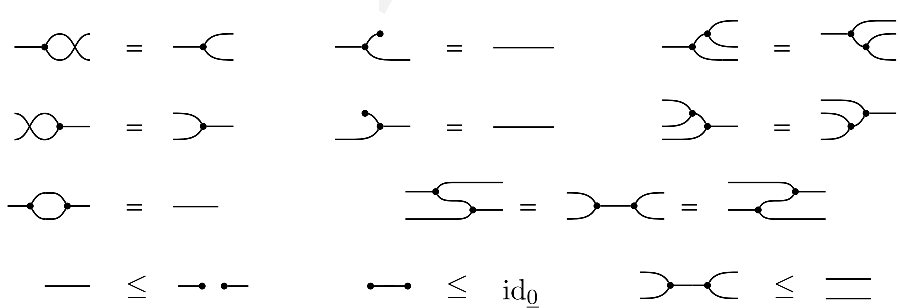
for all morphisms $\omega \colon \underline{n} \to \underline{m}$ in $\mathbb{W}$. □

Now that we have explicated the structure of the hom posets of $\mathbb{W}$, let us turn our attention to its morphisms, and for this purpose introduce a graphical notation prototypical of those to come. $\mathbb{W}$ may be generated by four morphisms, and we list these generating morphisms, their canonical cospan representatives in $\mathsf{Cspn}^{\mathrm{co}}$ (see Remark 3.7), and their graphical icons in the table below.

Morphism in $\mathbb{W}$	Corresponding cospan	Icon
$\epsilon \colon \underline{1} \to \underline{0}$	$\underline{1} \to \underline{1} \leftarrow \underline{0}$	
$\delta \colon \underline{1} \to \underline{2}$	$\underline{1} \to \underline{1} \leftarrow \underline{2}$	
$\eta \colon \underline{0} \to \underline{1}$	$\underline{0} \to \underline{1} \leftarrow \underline{1}$	
$\mu \colon \underline{2} \to \underline{1}$	$\underline{2} \to \underline{1} \leftarrow \underline{1}$	

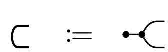
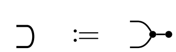
(3)

These generators satisfy the following equations and inequalities involving additionally the symmetry $\underline{2} \to \underline{2}$ of $\mathbb{W}$, and we render these constraints graphically with composition indicated via horizontal juxtaposition and tensor indicated via vertical juxtaposition.

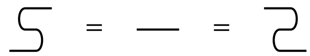


(4)

We refer to the composites $\eta \fatsemi \delta$ and $\mu \fatsemi \epsilon$ as the **cup** and the **cap**; they are denoted cup and cap and are depicted as follows:

 and  (5)

It follows from (4) that cap and cup satisfy the "yanking" or adjunction identities


(6)

The equations in the first and second lines of (4) are known as the **(co)commutativity, (co)unitality, and (co)associativity** equations for comonoids and monoids, respectively. The equations in the next line are known as the **special law** and the **frobenius law**. We refer to the inequalities in the last line as the **adjunction inequalities**, because they show up as the unit and co-unit of adjunctions, as we see next in the following proposition.

Proposition 3.9. *With notation as in* (3), *there are adjunctions*

$$\underline{1} \underset{\eta}{\overset{\epsilon}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \underline{0} \qquad \textit{and} \qquad \underline{1} \underset{\mu}{\overset{\delta}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \underline{2}\,. \tag{7}$$

Proof. The inequalities $\mathrm{id}_{\underline{1}} \leq (\epsilon \mathbin{;} \eta)$, $(\eta \mathbin{;} \epsilon) \leq \mathrm{id}_{\underline{1}}$, $\mathrm{id}_{\underline{2}} \leq (\mu \mathbin{;} \delta)$, and the equation $\mathrm{id}_{\underline{1}} = (\delta \mathbin{;} \mu)$ are all shown in (4), which itself is proved via computations in $\mathbb{C}\mathsf{spn}^{\mathrm{co}}$. The required equalities (1) are automatic in a po-category. □

Remark 3.10. The perhaps surprising half of the "special law", i.e. the inequality $(\delta \mathbin{;} \mu) \leq \mathrm{id}_{\underline{1}}$ not arising from adjointness, is in fact derivable from the rest of the structure:

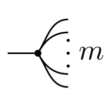
More generally, if $f \colon \underline{m} \twoheadrightarrow \underline{n}$ is any surjective function then $f^\dagger \mathbin{;} f = \mathrm{id}$, where $f^\dagger$ is the transpose (left adjoint) of f.

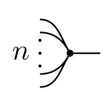
Definition 3.11 (Subprops of $\mathbb{W}$)**.** The prop $\mathbb{W}$ contains several other important full subprops:

- That generated by ϵ and δ is called the **prop for cocommutative comonoids**.
- That generated by η and μ is called the **prop for commutative monoids**.
- That generated by cup and cap (5) is called the **prop for self-duals**.

In fact these three props are equivalent to the monoidal category of finite sets, its opposite, and the category of unoriented cobordisms, respectively. See also [FS19c, Section 3].

Notation 3.12 (General arity δ and μ)**.** We will adopt the convention that for $n \in \mathbb{N}$ we will write $\delta^n \colon \underline{1} \to \underline{n}$ and $\mu^n \colon \underline{n} \to \underline{1}$ for the maps associated to the cospans $\underline{1} \to \underline{1} \leftarrow \underline{n}$ and $\underline{n} \to \underline{1} \leftarrow \underline{1}$ respectively. In particular we have $\delta^0 = \epsilon$, $\mu^0 = \eta$, and $\delta^1 = \mu^1 = \mathrm{id}_{\underline{1}}$ among other identities. Graphically we might render these morphisms as



and


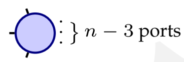
for δ^m and μ^n respectively.

3.2 Wiring diagrams for $\mathbb{W}$

The po-category $\mathbb{W}$ forms the foundation of our diagrammatic language for regular logic. We have already seen that morphisms in $\mathbb{W}$ can be given a graphical description by depicting generating morphisms using special icons, and working in the usual Joyal–Street string diagram language for morphisms in symmetric monoidal categories.

We now present an alternate way to depict morphisms in $\mathbb{W}$; we call these *wiring diagrams for* $\mathbb{W}$. In this section we begin our exploration of graphical regular logic by giving an explicit description of the objects, morphisms, 2-morphisms, and composition in $\mathbb{W}$ in terms of wiring diagrams for $\mathbb{W}$.

Notation 3.13 (Objects as shells)**.** By definition, an object $\underline{n} \in \mathbb{W}$ is a finite set. We represent it graphically by a circle with n ports around the exterior.

$\underline{n} =$  (8)

Our convention will be for the ports to be numbered clockwise from the left of the circle, unless otherwise indicated. We refer to such an annotated circle as a **shell**.

Definition 3.14. A **wiring diagram for** $\mathbb{W}$ is a morphism $\omega \colon \underline{n}_1 + \cdots + \underline{n}_k \to \underline{n}_{\text{out}}$.

Notation 3.15 (Graphical wiring diagrams for $\mathbb{W}$)**.** Suppose we have a wiring diagram $\omega \colon \underline{n}_1 + \cdots + \underline{n}_k \to \underline{n}_{\text{out}}$ in $\mathbb{W}$. Recall that such a morphism admits a canonical representation as a cospan of finite sets

$$\underline{n}_1 + \cdots + \underline{n}_k \xrightarrow{[\omega_1,\ldots,\omega_k]} \underline{n}_\omega \xleftarrow{\omega_{\text{out}}} \underline{n}_{\text{out}} \tag{9}$$

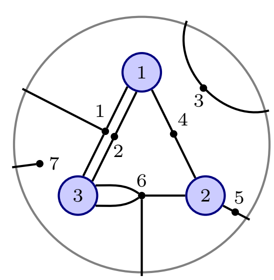
by Remark 3.7. With this in mind, we depict ω as follows.

1. Draw the shell for $\underline{n}_{\text{out}}$.
2. Draw each object $\underline{n}_i$, for $i = 1, \ldots, k$, as non-overlapping shells inside the $\underline{n}_{\text{out}}$ shell.
3. For each $i \in \underline{n}_\omega$, draw a black dot anywhere in the region interior to the $\underline{n}_{\text{out}}$ shell but exterior to all the $\underline{n}_i$ shells.
4. For each element $(i, j) \in \sum_{i=1,\ldots,k,\text{out}} \underline{n}_i$, draw a wire connecting the jth port on the object $\underline{n}_i$ to the black dot $\omega_i(j)$.

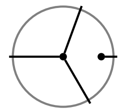
For a more compact notation, we may also neglect to explicitly draw the object $\underline{n}_{\text{out}}$, leaving it implicit as comprising the wires left dangling on the boundary of the diagram.

Example 3.16. Here is the combinatorial data of a wiring diagram $\omega \colon \underline{n}_1 + \underline{n}_2 + \underline{n}_3 \to \underline{n}_{\text{out}}$ in $\mathbb{W}$, together with its depiction:

$n_1 = 3, \quad n_2 = 3, \quad n_3 = 4, \quad n_{\text{out}} = 6, \quad n_\omega = 7$
$\omega_1(1) = 4, \omega_1(2) = 2, \omega_1(3) = 1,$
$\omega_2(1) = 6, \omega_2(2) = 4, \omega_2(3) = 5,$
$\omega_3(1) = 1, \omega_3(2) = 2, \omega_3(3) = \omega_3(4) = 6,$
$\omega_{\text{out}}(1) = 1, \omega_{\text{out}}(2) = \omega_{\text{out}}(3) = 3$
$\omega_{\text{out}}(4) = 5, \omega_{\text{out}}(5) = 6, \omega_{\text{out}}(6) = 7.$



Example 3.17. Note that we may have $k = 0$, in which case there are no inner shells. For example, the following has $n_\omega = 2$.



Remark 3.18. When multiple wires meet at a point, our convention will be to draw a dot iff the number of wires is different from two.

When wires intersect and we do not draw a black dot, the intended interpretation is that the wires are *not connected*:  $\neq$  .

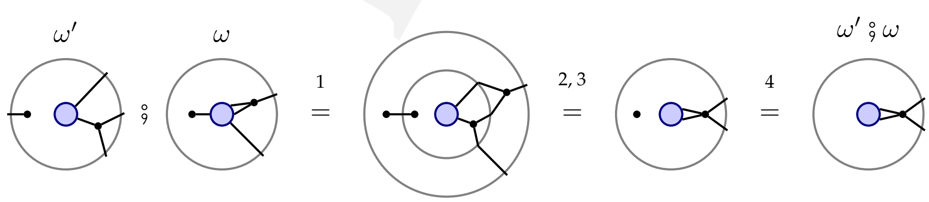
The following examples give a flavor of how composition, monoidal product, and 2-morphisms are represented using this graphical notation.

Example 3.19 (Composition as substitution). Composition of morphisms is described by **nesting** of wiring diagrams. Let $\omega' : \underline{n}' \to \underline{n}_1$ and $\omega : \underline{n}_1 \to \underline{n}_{\text{out}}$ be morphisms in $\mathbb{W}$. Then the composite relation $\omega' \mathbin{⨟} \omega : \underline{n}' \to \underline{n}_{\text{out}}$ is given by

1. drawing the wiring diagram for ω' inside the inner circle of the diagram for ω,
2. erasing the shell representing $\underline{n}_1$,
3. amalgamating any connected black dots into a single black dot,
4. removing either
   - (i) all but one of the black dots not connected to a shell (if $n' = n_{\text{out}} = 0$) or
   - (ii) all black dots not connected to a shell (if $n' \neq 0$ or $n_{\text{out}} \neq 0$).

Note that step 3 corresponds to taking pushouts in FinSet, while step 4 corresponds to taking the poset reflection.

As a shorthand for composition, we simply draw one wiring diagram directly substituted into another, as per step 1. For example, we have

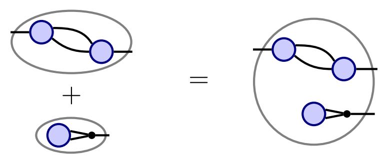


For the more general k-ary or operadic case, we may obtain the composite

$$(\underline{n}_1 + \cdots + \underline{n}_{i-1} + \omega' + \underline{n}_{i+1} + \cdots + \underline{n}_k) \mathbin{⨟} \omega$$

of any two morphisms $\omega' : \underline{n}'_1 + \cdots + \underline{n}'_k \to \underline{n}_i$ and $\omega : \underline{n}_1 + \cdots + \underline{n}_k \to \underline{n}_{\text{out}}$, with $1 \leq i \leq k$, by substituting the wiring diagram for ω' into the i^{th} inner circle of the diagram for ω, and following a procedure similar to that in Example 3.19.

Example 3.20 (Monoidal product as juxtaposition). Recall Corollary 3.8: the monoidal product of any morphism ω with $\eta \,⨾\, \epsilon$ or id_0 is again ω. In our graphical notation, if neither morphism is equal to $\eta \,⨾\, \epsilon\colon 0 \to 0$, then the monoidal product of two morphisms in $\mathbb{W}$ is simply their juxtaposition. For example, we might have:



Example 3.21 (2-morphisms as breaking wires and removing disconnected black dots). Let $\omega, \omega' \colon \underline{n} \to \underline{n}_{\text{out}}$ be morphisms in $\mathbb{W}$. Each is canonically represented by a cospan of finite sets,

$$\underline{n} \to \underline{n}_{\omega} \leftarrow \underline{n}_{\text{out}} \qquad \text{and} \qquad \underline{n} \to \underline{n}_{\omega'} \leftarrow \underline{n}_{\text{out}} .$$

By definition, there exists a 2-morphism $\omega \leq \omega'$ iff there is a function $x\colon \underline{n}_{\omega'} \to \underline{n}_{\omega}$ making the requisite diagrams commute. For any element $i \in \underline{n}_{\omega}$, the pre-image $x^*(i)$ is either empty, has one element, or has multiple elements. In the first case, the pair of wiring diagrams depicting each side of the inequality $\omega \leq \omega'$ would show dot i being removed; in the second case, it would show dot i remaining as it was; and in the third case, it would show a connection being broken at dot i. For example, we have 2-morphisms  $\leq$  and  $\leq$ .

3.3 Supply

It often happens that every object in a symmetric monoidal category $\mathcal{C}$ is equipped with the same sort of algebraic structure – say coming from a prop $\mathbb{P}$ – with the property that these algebraic structures are compatible with the monoidal structure. In [FS19c], we refer to this situation by saying that $\mathcal{C}$ *supplies* $\mathbb{P}$. For our purposes we need to slightly generalize this theory, from props to po-props and from symmetric monoidal categories $\mathcal{C}$ to symmetric monoidal po-categories $\mathbb{C}$.

Definition 3.22 (Supply)**.** Let $\mathbb{P}$ be a po-prop and $\mathbb{C}$ a symmetric monoidal po-category. A **supply of** $\mathbb{P}$ **in** $\mathbb{C}$ consists of a strong monoidal po-functor $s_c\colon \mathbb{P} \to \mathbb{C}$ for each object $c \in \mathbb{C}$, such that

(i) $s_c(m) = c^{\otimes m}$ for each $m \in \mathbb{N}$,

(ii) the strongator $c^{\otimes m} \otimes c^{\otimes n} \to c^{\otimes (m+n)}$ is equal to the associator for each $m, n \in \mathbb{N}$,

(iii) the following diagrams commute for every $c, d \in \mathbb{C}$ and $\mu\colon m \to n$ in $\mathbb{P}$, where the σ's are the symmetry isomorphisms from (2).

$$\begin{array}{ccc} c^{\otimes m} \otimes d^{\otimes m} & \xrightarrow{s_c(\mu) \otimes s_d(\mu)} & c^{\otimes n} \otimes d^{\otimes n} \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ (c \otimes d)^{\otimes m} & \xrightarrow[s_{c \otimes d}(\mu)]{} & (c \otimes d)^{\otimes n} \end{array} \qquad \begin{array}{ccc} I & = & I \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ I^{\otimes m} & \xrightarrow[s_I(\mu)]{} & I^{\otimes n} \end{array} \tag{10}$$

We often denote the morphism $s_c(\mu)$ in $\mathbb{C}$ simply by $\mu_c \colon c^{\otimes m} \to c^{\otimes n}$ for typographical reasons; i.e. we elide explicit mention of s.

We further say that $f \colon c \to d$ in $\mathbb{C}$ is a **lax s-homomorphism** (resp. **oplax s-homomorphism**) if, for each $\mu \colon m \to n$ in the prop $\mathbb{P}$, there is a 2-morphism as shown in the left-hand (resp. right-hand) diagram:

$$\begin{array}{ccc} c^{\otimes m} & \xrightarrow{f^{\otimes m}} & d^{\otimes m} \\ {\scriptstyle \mu_c}\downarrow & \geq & \downarrow{\scriptstyle \mu_d} \\ c^{\otimes n} & \xrightarrow[f^{\otimes n}]{} & d^{\otimes n} \end{array} \qquad\qquad \begin{array}{ccc} c^{\otimes m} & \xrightarrow{f^{\otimes m}} & d^{\otimes m} \\ {\scriptstyle \mu_c}\downarrow & \leq & \downarrow{\scriptstyle \mu_d} \\ c^{\otimes n} & \xrightarrow[f^{\otimes n}]{} & d^{\otimes n} \end{array} \quad .$$

Since $\mathbb{C}$ is locally posetal, if f is both a lax and an oplax s-homomorphism, then these diagrams commute and we simply say f is an **s-homomorphism**.

We say that $\mathbb{C}$ **(lax-/oplax-) homomorphically supplies** $\mathbb{P}$ if every morphism f in $\mathbb{C}$ is a (lax/oplax) s-homomorphism.

Example 3.23. An important class of examples of homomorphic supply are those categories with finite products. By the main theorem of [Fox76], replicated below, such categories are precisely the discretely ordered po-categories that homomorphically supply cocommutative comonoids (Definition 3.11).

Proposition 3.24. *A category $\mathcal{C}$ has finite products iff it can be equipped with a homomorphic supply of commutative comonoids. If $\mathcal{C}$ and $\mathcal{D}$ have finite products, a functor $\mathcal{C} \to \mathcal{D}$ preserves them iff it preserves the supply of comonoids.*

Example 3.25. We shall meet another large class of examples of lax homomorphic supply in Section 6, wherein we shall find that regular categories are equivalently po-categories with a lax homomorphic supply of $\mathbb{W}$ and some additional structure.

Now that we have established the definition of supply we collect some results which we will variously leverage in our later sections.

Notation 3.26 (Coproduct of symmetric monoidal po-categories)**.** We will write $\bigsqcup$ to denote the coproduct of symmetric monoidal po-categories in the 2-category SMC of symmetric monoidal po-categories, symmetric monoidal po-functors, and monoidal natural transformations.

Warning 3.27. The coproduct of symmetric monoidal categories in SMC *does not* coincide with the po-categorical coproduct in $\mathcal{P}\mathsf{oCat}$. Instead, for a set J and symmetric monoidal po-categories $\{\mathbb{C}_j\}_{j\in J}$,

$$\operatorname{Ob}\Big(\coprod_{j\in J}\mathbb{C}_j\Big) := \Big\{(c_j) \in \prod_{j\in J}\operatorname{Ob}\mathbb{C}_j \;\Big|\; c_j = I_j \text{ for all but finitely many } j \in J\Big\}\,.$$

See [FS19c, Theorem 2.2 & Appendix A] for details.

Lemma 3.28. *Let $\mathbb{C}$ be a symmetric monoidal po-category supplying $\mathbb{P}$ a po-prop. Then for any set J, the supply s of $\mathbb{P}$ in $\mathbb{R}$ extends to a supply $\tilde{s}$ of $\mathbb{C}$ in $\coprod_J \mathbb{C}$ such that*

$$\tilde{s}_{(c_j)_J} \mathbin{;} \pi_i = \begin{cases} I, & c_i = I \\ s_{c_i}, & \text{otherwise} \end{cases}\,,$$

as functors $\mathbb{P} \to \mathbb{C}$. If in particular $\mathbb{C}$ is symmetric strict monoidal then $\tilde{s}$ satisfies $\tilde{s}_{(c_j)_j} \mathbin{;} \pi_i = s_{c_i}$.

Proof. The supply conditions (Definition 3.22) hold "point-wise" for each c_i, and as the symmetries of $\coprod_J \mathbb{C}$ are point-wise those of $\mathbb{C}$, the functors $\tilde{s}_{(c_i)_I}$ constitute a supply of $\mathbb{P}$ in $\coprod \mathbb{C}$. □

In what follows, we do not produce proofs for Propositions 3.29 to 3.31 as these were essentially proven in [FS19c, Propositions 3.13, 3.14, and 3.21]; the change from props to po-props makes no difference in this context.

Proposition 3.29. *A supply s of $\mathbb{P}$ in $\mathbb{C}$ induces a strong monoidal po-functor $s^{\sqcup}\colon \coprod_{\operatorname{Ob}\mathbb{C}} \mathbb{P} \to \mathbb{C}$ uniquely determined by $\iota_c \mathbin{;} s^{\sqcup} = s_c$ for each $c \in \mathbb{C}$ and inclusion $\iota_c\colon \mathbb{P} \rightarrowtail \coprod_{\operatorname{Ob}\mathbb{C}} \mathbb{P}$.* □

Proposition 3.30. *Let $\mathbb{P}$ be a po-prop. Then there is a supply of $\mathbb{P}$ in $\mathbb{P}$ where the functors s_c for $c \in \operatorname{Ob}\mathbb{P}$ and $\mu\colon m \to n$ of $\mathbb{P}$ are given by*

$$s_c(\mu)\colon s_c(m) = \underbrace{c + \ldots + c}_{m} = \underbrace{m + \ldots + m}_{c} \xrightarrow{\mu+\ldots+\mu} \underbrace{n + \ldots + n}_{c} = \underbrace{c + \ldots + c}_{n} = s_c(n)\,. \quad □$$

Proposition 3.31 (Change of supply)**.** *Let $G\colon \mathbb{P} \to \mathbb{Q}$ be a po-prop functor. For any supply s of $\mathbb{Q}$ in $\mathbb{C}$, we have a supply $(G \mathbin{;} s)$ of $\mathbb{P}$ in $\mathbb{C}$.* □

Example 3.32. Any supply of $\mathbb{W}$ in $\mathbb{C}$ induces a supply of cocommutative comonoids in $\mathbb{C}$ by change of supply and Definition 3.11.

Example 3.33. A symmetric monoidal po-category $(\mathbb{C}, I, \otimes)$ is self-dual compact closed iff it supplies self duals, cup and cap (see Definition 3.11). Thus $\mathbb{W}$ is self-dual compact closed by Proposition 3.30, as in any po-category $\mathbb{C}$ supplying $\mathbb{W}$ by Proposition 3.31.

Example 3.34. Consider the po-prop of finite sets and co-relations, where a morphism $m \to n$ is an equivalence relation on $m + n$, and the order is given by coarsening. Since this po-prop receives a po-prop functor from $\mathbb{W}$ (see Proposition 3.5), it supplies $\mathbb{W}$ by Propositions 3.30 and 3.31.

Definition 3.35 (Preservation of supply)**.** Let $\mathbb{P}$ be a po-prop, $\mathbb{C}$ and $\mathbb{D}$ symmetric monoidal po-categories, and suppose s is a supply of $\mathbb{P}$ in $\mathbb{C}$ and t is a supply of $\mathbb{P}$ in $\mathbb{D}$. We say that a strong symmetric monoidal po-functor $(F, \varphi)\colon \mathbb{C} \to \mathbb{D}$ **preserves the supply** if the strongators φ provide an isomorphism $t_{Fc} \cong (s_c \mathbin{⨟} F)$ of po-functors $\mathbb{P} \to \mathbb{D}$ for each $c \in \mathbb{C}$.

Unpacking, a strong monoidal po-functor (F, φ) preserves the supply iff the following diagram commutes for each morphism $\mu\colon m \to n$ in $\mathbb{P}$ and object $c \in \mathbb{C}$:

$$\begin{array}{ccc} F(c)^{\otimes m} & \xrightarrow{\mu_{F(c)}} & F(c)^{\otimes n} \\ \varphi\big\downarrow & & \big\downarrow\varphi \\ F(c^{\otimes m}) & \xrightarrow[F(\mu_c)]{} & F(c^{\otimes n}) \end{array} \tag{11}$$

As we might expect, the most abundant example of preservation of supply arises from the supply of a po-prop P in itself.

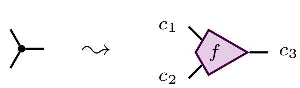
Lemma 3.36. *Let $\mathbb{C}$ be a po-category supplying $\mathbb{P}$ a po-prop, and let $c \in \operatorname{Ob}\mathbb{C}$ be an object. Then the strong symmetric monoidal po-functor $s_c\colon \mathbb{P} \to \mathbb{C}$ determined by the supply of $\mathbb{P}$ in $\mathbb{C}$ preserves the supply of $\mathbb{P}$ in itself of Proposition 3.30.*

Proof. By unwinding this claim we may see that it reduces to iterated applications of the supply conditions (10) for s_c. □

3.4 General wiring diagrams

The notion of supply will enable us to extend our graphical notation of Section 3.2 for morphisms in $\mathbb{W}$ to graphical notation for morphisms in any po-category $\mathbb{C}$ supplying $\mathbb{W}$. The extension of our graphical notation is to the general notion of *wiring diagram for* $\mathbb{C}$. Since $\mathbb{W}$ supplies $\mathbb{W}$ by Proposition 3.30, we will be able to verify that the current material is a generalisation of the previous.

In the presence of objects and morphisms of a po-category $\mathbb{C}$ supplying $\mathbb{W}$, the change in our notation is essentially that each wire is labelled by an object in $\mathbb{C}$ and nodes may be replaced by any morphism in $\mathbb{C}$:



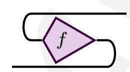
Note that we have used two related methods for depicting morphisms in $\mathbb{W}$. The first is the string diagrams of Joyal–Street, which we used in Section 3.1. In these diagrams, the domain of the morphism is represented on the left of the diagram, and the codomain is on the right. The second method is the wiring diagrams of Section 3.2. For these diagrams, the domain of the morphism is represented by the interior blue circles, while the codomain is represented by the outer circle. Similarly, we will use two analogous methods for depicting morphisms in a po-category supplying $\mathbb{W}$. We first discuss the Joyal–Street string diagrams.

String diagrams in a po-category supplying $\mathbb{W}$

As we shall soon see, in a po-category supplying $\mathbb{W}$ we may bijectively move tensor factors of the domain to the codomain and visa-versa by making use of the supplied images of the morphisms `cup` and `cap`. Roughly speaking, this allows us to treat the distinction between domain and codomain as a matter of bookkeeping. We give two important examples of this, and depict them in string diagrams.

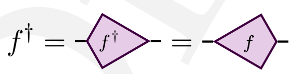
The first is the transpose of a morphism, which exchanges domain and codomain.

Definition 3.37 (Transpose)**.** Let $f\colon c \to d$ be a morphism in a self dual compact closed po-category $\mathbb{C}$. We define its **transpose** $f^\dagger\colon d \to c$ to be the morphism

$f^\dagger :=$ 

Notation 3.38 (Transpose)**.** Note that we have used the asymmetric icon f to denote f. This asymmetry allows us to distinguish the domain against the codomain without appealing to the orientation of the picture: the domain c is attached at largest angle of the kite, while the domain d is attached at the least angle.

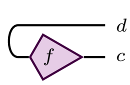
In line with this convention, and the adjunction equations (6), we depict the transpose simply by the 180-degree rotation of f:

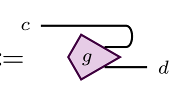


We will make liberal use of asymmetric icons when it will help to have rotation denote transpose.

Transposition gives an isomorphism $\mathbb{C}(c,d) \cong \mathbb{C}(d,c)$ of posets. The second example of moving factors between domain and codomain is given by naming and unfolding. These give an isomorphism $\mathbb{C}(c,d) \cong \mathbb{C}(I, c \otimes d)$.

Definition 3.39 (Name and unfolding)**.** Let $f\colon c \to d$ and $g\colon I \to c \otimes d$ be morphisms in a self dual compact closed po-category. We define the **name** $f^{\sqsubset}\colon I \to c \otimes d$ of f and the **unfolding** $g^{\sqsupset}\colon c \to d$, to be respectively:

$f^{\sqsubset} :=$ 

$g^{\sqsupset} :=$ 

$\mathtt{cup}_c \fatsemi (\mathrm{id}_c \otimes f)$ $\qquad$ $\rho_c^{-1} \fatsemi (\mathrm{id}_c \otimes g) \fatsemi (\mathtt{cap}_c \otimes \mathrm{id}_d) \fatsemi \lambda_d$

We invite the reader to construct the pleasing graphical proof of the following.

Lemma 3.40. *The maps $\cdot^{\sqsubset}$ and $\cdot^{\sqsupset}$ are mutually inverse.* $\square$

Recall that we refer to the generating morphisms of $\mathbb{W}$ as $(\eta, \mu, \epsilon, \delta)$; see (3). The equations in (4) imply that $(\eta, \mu, \epsilon, \delta)$ form a *special commutative frobenius monoid*. The theory of special commutative frobenius monoids is given by the prop $\mathsf{Cspn}(\mathsf{FinSet})$ of finite sets and cospans between them [Lac04]. Moreover, if we consider the prop $\mathsf{Cspn}(\mathsf{FinSet})$ as a 2-discrete po-prop, there is a po-prop functor $\mathsf{Cspn}(\mathsf{FinSet}) \to \mathbb{W}$ taking a cospan to the morphism of $\mathbb{W}$ that it represents. Thus if $\mathbb{C}$ supplies $\mathbb{W}$ then by Proposition 3.31 we see that $\mathbb{C}$ supplies special commutative frobenius monoids, and the underlying 1-category of $\mathbb{C}$ is what is known as a **hypergraph category** [FS19a].

Hypergraph structure implies that we may use network or circuit-like diagrams to represent 1-morphisms: the additional structures and axioms of a hypergraph category allow us to split and combine wires in various ways, but such that "connectivity is all that matters" when interpreting string diagrams. This allows us to be rather informal when depicting morphisms as string diagrams – in particular how a given connection is constructed from frobenius maps – because all formalisations of it result in the same morphism.

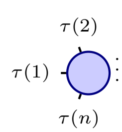
For example, the following diagrams all canonically represent the same morphism:

We will make extensive use of this notation, and refer the reader unfamiliar with it to [FS19a] for more details.

Wiring diagrams in a po-category supplying $\mathbb{W}$

Having considered the Joyal-Street-type string diagrams in a po-category supplying $\mathbb{W}$, we are now ready to extend our notation of Section 3.2 to this context.

Definition 3.41 (Shell)**.** Let $\mathbb{C}$ be a po-category supplying $\mathbb{W}$. A **shell** $\Gamma = (n, \tau)$ in $\mathbb{C}$ is a function $\tau \colon \underline{n} \to \operatorname{Ob}\mathbb{C}$. We depict a shell as a circle with n ports, labelled clockwise starting from its left with the objects $\tau(i) \in \operatorname{Ob}\mathbb{C}$:

$\Gamma =$ 

We shall abuse notation by writing Γ for both the pair (n, τ) as well as the tensor product $\Gamma \coloneqq \tau(1) \otimes \cdots \otimes \tau(n)$.

Definition 3.42 (Wiring diagram)**.** Let $\mathbb{C}$ be a po-category supplying $\mathbb{W}$. A **wiring diagram for** $\mathbb{C}$ is:

(i) a natural number k,

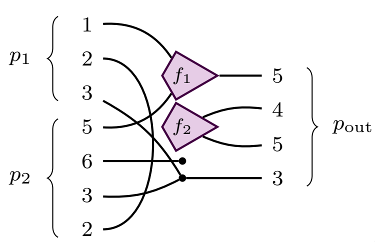
(ii) for each $i \in \{1, \dots, k, \text{out}\}$, a shell $\Gamma_i := (n_i, \tau_i)$,

(iii) a morphism $\omega\colon \Gamma_1 \otimes \cdots \otimes \Gamma_k \to \Gamma_{\text{out}}$ in $\mathbb{C}$; that is, a morphism

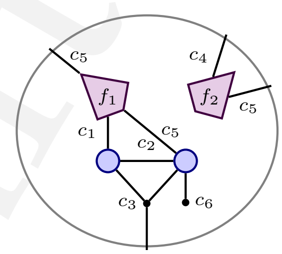
$$\bigotimes_{i=1,\dots,k} \bigotimes_{j=1,\dots,n_k} \tau_i(j) \xrightarrow{\quad\omega\quad} \bigotimes_{\ell=1,\dots,n_{\text{out}}} \tau_{\text{out}}(\ell)$$

Notation 3.43 (Wiring diagrams for a supply)**.** We depict wiring diagrams for $\mathbb{C}$ in much the same way as we depict wiring diagrams for $\mathbb{W}$, except that wires are now labelled by objects of $\mathbb{C}$, and dots connecting wires may now be drawn as kites labelled by morphisms of $\mathbb{C}$ or remain as dots to represent morphisms of $\mathbb{C}$ in the supply for $\mathbb{W}$.

Example 3.44. Let $p_1 := (c_1 \otimes c_2 \otimes c_3)$, $p_2 := (c_5 \otimes c_6 \otimes c_3 \otimes c_2)$, and $p_{\text{out}} := c_5 \otimes c_4 \otimes c_5 \otimes c_3$, and suppose given $f_1\colon c_1 \otimes c_5 \to c_5$ and $f_2\colon I \to c_4 \otimes c_5$ in $\mathbb{C}$. Using the supply of $\mathbb{W}$ we also have the co-unit $\varepsilon_{c_6}\colon c_6 \to I$, $\mathtt{cap}_{c_2}\colon c_2 \otimes c_2 \to I$, and the comultiplication $\delta_{c_3}\colon c_3 \otimes c_3 \to c_3$ in $\mathbb{C}$. Together these provide a morphism $p_1 \otimes p_2 \to p_{\text{out}}$ as shown below-left, which we can render as the wiring diagram for $\mathbb{C}$ below-right.



String diagram



Wiring diagram

In the coming sections, where it causes no ambiguity, we will gradually suppress infer-able details in our wiring diagrams.

4 Regular calculi

Graphical regular logic is the graphical representation of the logical calculus carried by *regular caluli*, those objects understanding contexts, predicates, and the regular fragment of logic thereupon. In this section we will introduce the framework of *right ajax po-functors* which will allow us to compactly capture all of the desired properties of regular calculi. We will then define regular calculi in terms of these right ajax po-functors and assemble regular calculi into a suitable 2-category.

4.1 Right ajax po-functors and right adjoint monoids

Definition 4.1 (Right ajax po-functor)**.** Let $\mathbb{C}$ and $\mathbb{D}$ be symmetric monoidal po-categories. A **right adjoint-lax** or **right ajax** po-functor $F\colon \mathbb{C} \to \mathbb{D}$ is a lax symmetric monoidal po-functor for which the laxators are right adjoints.

We denote the laxators by ρ and their left adjoints by λ:

$$I \underset{\lambda}{\overset{\rho}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} F(I) \qquad \text{and} \qquad F(c) \otimes F(c') \underset{\lambda_{c,c'}}{\overset{\rho_{c,c'}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} F(c \otimes c') \ . \tag{12}$$

Example 4.2 (Cartesian monoidal po-categories)*.* If $\mathbb{D}$ is a cartesian monoidal po-category then every symmetric monoidal po-category $\mathbb{C}$ has a canonical right ajax po-functor $\mathbb{C} \to \mathbb{D}$, viz., the constant po-functor at the terminal object of $\mathbb{D}$. This po-functor is easily seen to be terminal in the po-category of right ajax po-functors $\mathbb{C} \to \mathbb{D}$ and monoidal 2-natural transformations. Later, in Lemma 7.9 we shall determine, for a class of symmetric monoidal po-categories $\mathbb{C}$, the initial right ajax po-functor $\mathbb{C} \to \mathbb{P}\mathsf{oset}$. Once we have proven Proposition 4.6 below we will see that this is a special case of the composition of po-functors $\mathbb{C} \to 1 \to \mathbb{D}$ and the fact that 1 is an *right adjoint monoid* in $\mathbb{D}$.

Example 4.3 (Represented po-functors)*.* As we shall see later in Sections 4.2 and 6.3 there are many important classes of examples of right ajax po-functors. One such class, established in Proposition 6.22, is the represented po-functor $\mathbb{R}(I, -) \colon \mathbb{R} \to \mathbb{P}\mathsf{oset}$ for any *prerelational* po-category $\mathbb{R}$. Examples of prerelational po-categories include the po-category $\mathsf{LAdj}\,\mathcal{R}$ of left adjoints in a regular category $\mathbb{R}$ – see Theorem 6.19 for details.

Warning 4.4*.* If F is a right ajax po-functor, the left adjoints $\lambda_{c,c'}$ of the laxators *do not* in general equip F with the structure of an oplax monoidal functor. *A priori*, the naturality squares for the laxators ρ are only lax-naturality squares for their left adjoints λ.

Let us record two immediate but nevertheless useful consequences of the definition of right ajax po-functors.

Lemma 4.5. *Every strong monoidal functor between monoidal po-categories is right ajax. The composite of right ajax po-functors is again right ajax.*

To understand right ajax po-functors and their importance in the story of regular logic, it is useful to introduce the notion of *right adjoint monoid*. Let us write 1 for the terminal monoidal po-category.

Proposition 4.6. *Let $(\mathbb{C}, I, \otimes)$ be a symmetric monoidal po-category. There is a bijection between:*

1. *The set of right ajax po-functors $1 \to \mathbb{C}$,*
2. *The set of commutative monoid objects (c, μ, η) such that μ and η are right adjoints,*
3. *The set of cocommutative comonoid objects (c, δ, ϵ) such that δ and ϵ are left adjoints.*

Proof. (1) $\Leftrightarrow$ (2): The set $\mathsf{Lax}(1, \mathbb{C})$ of lax symmetric monoidal po-functors $1 \to \mathbb{C}$ may be seen to be in bijection with the set of commutative monoid objects (c, μ, η) in $\mathbb{C}$. Indeed, η and unit μ come from the 0-ary and 2-ary laxators respectively: $\eta = \rho$ and $\mu = \rho_{1,1}$. Hence the added condition that η and μ have left adjoints is precisely the right ajax condition.

(2) ⇔ (3): This bijection is implemented by taking adjoints. That is, given an object $c \in \mathbb{C}$ and two adjunctions

$$I \underset{\epsilon}{\overset{\eta}{\underset{\top}{\rightleftarrows}}} c \qquad \text{and} \qquad c \otimes c \underset{\delta}{\overset{\mu}{\underset{\top}{\rightleftarrows}}} c\ , \tag{13}$$

it may be verified that μ and η satisfy the commutative monoid laws if and only if δ and ϵ satisfy the cocommutative comonoid laws. □

To summarize, if $(c, \rho, \lambda) \colon 1 \to \mathbb{C}$ is a right ajax po-functor then the corresponding monoid and comonoid structures on c are given by

$$\eta = \rho \qquad \mu = \rho_{1,1} \qquad \text{and} \qquad \epsilon = \lambda \qquad \delta = \lambda_{1,1} \tag{14}$$

Proposition 4.6 motivates the following definition.

Definition 4.7 (Right adjoint monoid)**.** Let $(\mathbb{C}, I, \otimes)$ be a monoidal po-category. A **right adjoint monoid** in $\mathbb{C}$ is a commutative monoid object (c, μ, η) in $\mathbb{C}$ such that μ and η are right adjoints.

Right adjoint monoids are a slight weakening of *internal* $\wedge$*-semilattices*; see [Sch94, Chapter 5] and references therein. To get a feel for why this might be so, it helps to first recall the po-categorical version of a well-known lemma.

Lemma 4.8. *Let $\mathbb{C}$ be a monoidal po-category. If the monoidal structure is cartesian (given by finite products in the underlying 1-category) then every object has a unique comonoid structure, and it is cocommutative.*

Proof. Since the unit object is terminal, the maps $c \times \epsilon$ and $\epsilon \times c$ are forced to be the projections $c \times c \to c$, so δ is forced to be the diagonal. Commutativity follows by universal property arguments. □

Example 4.9 (Right adjoint monoids in $\mathbb{P}$oset are $\wedge$-semilattices). A poset $P \in \mathbb{P}$oset is an adjoint monoid iff it is a meet-semilattice, in which case $\eta = \texttt{true}$ and $\mu = \wedge$. To see this we can use Lemma 4.8. This states that any poset P has a unique comonoid structure given by the terminal and diagonal maps $\epsilon \colon P \to 1$ and $\delta \colon P \to P \times P$. Thus P is an adjoint monoid iff these maps have right adjoints as in (13), which holds iff η is a top element and μ is a meet.

Example 4.10 (Right adjoint monoids in a relational po-category are objects). In later sections we will have cause to consider *relational* and *prerelational* po-categories. By Theorem 6.19, every relational po-category is equivalent to the po-category of left adjoints $\mathsf{LAdj}\,\mathcal{R}$ in a regular category $\mathcal{R}$. In particular, $\mathsf{LAdj}\,\mathcal{R}$ is cartesian monoidal and so Lemma 4.8 above applies to relational po-categories: each object has a unique cocommutative comonoid structure. By Proposition 4.6 (2) ⇔ (3) we see that each object in fact must have a unique commutative monoid structure.

It will be useful to know that, just as lax monoidal functors map monoids to monoids, right ajax po-functors map right adjoint monoids to right adjoint monoids.

Proposition 4.11. *Right ajax po-functors send right adjoint monoids to right adjoint monoids.*

Proof. The composite of right ajax po-functors is again right ajax, so the result follows from Proposition 4.6. ☐

Although this result may seem anodyne at present, we will consistently leverage this in our development of graphical regular calculus in Section 5 as it lends our graphical calculus its "regular" aspect.

4.2 Regular calculi as indexed right adjoint monoids

Now that we have understood the notion of right ajax po-functor, we move to define one of the central notions in this paper.

Definition 4.12 (Regular calculus)**.** A **regular calculus** P consists of a pair $(\mathbb{C}_P, P)$ where $(\mathbb{C}_P, I, \otimes)$ is a symmetric monoidal po-category supplying $\mathbb{W}$ and $P\colon \mathbb{C}_P \to \mathbb{P}\mathsf{oset}$ is a right ajax po-functor, whose laxators we denote by `true` and $\boxplus$:

$$1 \underset{\lambda_I}{\overset{\mathtt{true}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} P(I) \qquad \text{and} \qquad P(\Gamma_1) \times P(\Gamma_2) \underset{\lambda_{\Gamma_1,\Gamma_2}}{\overset{\boxplus_{\Gamma_1,\Gamma_2}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} P(\Gamma_1 \otimes \Gamma_2)\ . \tag{15}$$

We call $\mathbb{C}_P$ the po-category of **contexts**, and P the **predicates** po-functor. If $\Gamma \in \operatorname{Ob} \mathbb{C}_P$ is a context, and $\theta, \theta' \in P(\Gamma)$ are predicates in context Γ such that $\theta \leq \theta'$, then we say that θ **entails** θ', and write $\theta \vdash \theta'$. Finally, a regular calculus is termed **bare** if $\mathbb{C}_P = \bigsqcup_J \mathbb{W}$ is a coproduct[1] of copies of $\mathbb{W}$.

As a first example let us cast $\wedge$-semilattices as a regular calculi.

Example 4.13 ($\wedge$-semilattices are regular calculi)*.* Observe that the terminal po-category 1 supplies $\mathbb{W}$. By Proposition 4.6 and Example 4.9 a right ajax po-functor $1 \to \mathbb{P}\mathsf{oset}$ is the same as a $\wedge$-semilattice. Hence a regular calculus whose po-category of contexts is terminal is a $\wedge$-semillatice.

Should we fix a meet semi-lattice L viewed as a right ajax po-functor $L\colon 1 \to \mathbb{P}\mathsf{oset}$, we may extend this idea to endow any po-category $\mathbb{C}$ which supplies $\mathbb{W}$ with the structure of a regular calculus by considering the composite po-functor $\mathbb{C} \xrightarrow{!} 1 \xrightarrow{L} \mathbb{P}\mathsf{oset}$.

This phenomenon, that regular calculi induce $\wedge$-semilattice structures on their posets of predicates, is not special to factoring through the terminal po-category 1. In the following sense it must occurs point-wise for any regular calculus.

[1] see Warning 3.27

Proposition 4.14. *If* $(\mathbb{C}_P, P)$ *is a regular calculus then the poset* $P(\Gamma)$ *has the structure of a meet-semilattice for each* $\Gamma \in \mathbb{C}_P$*.*

Proof. Recall that by supplying $\mathbb{W}$, $\mathbb{C}_P$ has a chosen right adjoint monoid structure supplied for each object Γ: $(\eta_\Gamma, \mu_\Gamma)$. The result then follows from fact that ajax po-functors send right adjoint monoids to right adjoint monoids by Proposition 4.11, as well as Example 4.9. Explicitly, (12) and (7) give rise to the following adjunctions.

$$1 \underset{\lambda_I}{\overset{\texttt{true}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} P(I) \underset{P(\epsilon_\Gamma)}{\overset{P(\eta_\Gamma)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} P(\Gamma) \quad \text{and} \quad P(\Gamma) \times P(\Gamma) \underset{\lambda_{\Gamma,\Gamma}}{\overset{\boxplus_{\Gamma,\Gamma}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} P(\Gamma \otimes \Gamma) \underset{P(\delta_\Gamma)}{\overset{P(\mu_\Gamma)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} P(\Gamma) \qquad \square$$

This result motivates the following notation.

Notation 4.15 (Right adjoint monoids)**.** In a regular calculus $(\mathbb{C}_P, P)$, for each context $\Gamma \in \operatorname{Ob} \mathbb{C}_P$ we have an right adjoint monoid structure on $P(\Gamma)$ which we denote by

$$1 \underset{!}{\overset{\texttt{true}_\Gamma}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} P(\Gamma) \qquad \text{and} \qquad P(\Gamma) \times P(\Gamma) \underset{\Delta_\Gamma}{\overset{\wedge_\Gamma}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} P(\Gamma) \;. \tag{16}$$

In keeping with the conventions thus far, for $n \in \mathbb{N}$ we will write $\wedge^n_\Gamma$ for the composite map $\boxplus^n_\Gamma \mathbin{;} P(\mu^n_\Gamma)\colon P(\Gamma)^{\times n} \to P(\Gamma)$. In the case of $n = 0$ we see that $\wedge^0_\Gamma = \texttt{true}_\Gamma$, and we will freely confuse this map $\texttt{true}_\Gamma$ with the top element $\texttt{true}_\Gamma(*)$. Where it will cause no ambiguity we will omit the label Γ on the maps $\texttt{true}$, $\wedge$, $\wedge^n$, $\boxplus$, and so forth.

Further examples of regular calculi abound.

Example 4.16. In categorical logic, given a regular theory, we can construct a regular calculus where the category of contexts has contexts as objects, and where the predicates functor maps a context to the set of formulas in those variables.

Example 4.17. An important class of examples arise from the "prerelational" po-categories of Section 6.1. There we will establish that the represented right ajax po-functor $\mathbb{R}(I, -)$ of Example 4.3 above, for "prerelational" $\mathbb{R}$, gives $(\mathbb{R}, \mathbb{R}(I, -))$ the structure of regular calculus. This regular calculus has as contexts the objects r of $\mathbb{R}$, and as predicates in such a context morphisms $I \to r$. We call the assignment $\mathbb{R} \mapsto (\mathbb{R}, \mathbb{R}(I, -))$ "taking predicates", and Section 6.3 stands to establish its 2-functoriality. Ultimately we shall see that taking predicates is a certain type of 2-dimensional inclusion, and that this inclusion is part of a 2-dimensional reflection.

Example 4.18. As an example of the previous example, by Example 6.4 later one may take $\mathbb{R}$ to be the po-category of finite sets and corelations. The resulting regular calculus assigns has as contexts finite sets S, and predicates in such contexts are equivalently partitions of S.

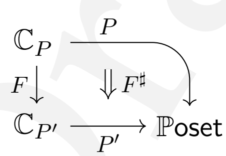
Example 4.19. Another example of Example 4.17 is given by regular categories. Since $\mathbb{R}\mathsf{el}\,\mathcal{R}$ gives a "prerelational" po-category for any regular category $\mathcal{R}$, we obtain a regular calculus for every regular category. Here the contexts are the objects r of $\mathcal{R}$, and the predicates in such contexts are equivalently the subobjects of r. By composing the 2-equivalence $\mathbb{R}\mathsf{el}$ with "taking predicates" we obtain similarly a 2-dimensional reflection of regular categories in regular calculi.

Example 4.20. Taking predicates is not the only way to extract a regular calculus from a "prerelational" po-category $\mathbb{R}$. In Proposition 7.2 later we will see how to construct a *bare* regular calculus from any prerelational po-category, and so in particular from any regular category. Unlike taking predicates, here the contexts are finite lists of objects $(r_1, \ldots, r_n)$ of $\mathbb{R}$, and the predicates in such contexts are equivalently morphisms whose domain $d_1 \otimes \cdots \otimes d_k$ and codomain $c_1 \otimes \cdots \otimes c_m$ tensored together give $\bigotimes r_i$.

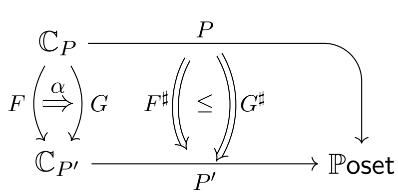
4.3 The 2-category $\mathcal{R}\mathsf{gCalc}$ of regular calculi

The central result of the companion paper compares the 2-category theory of regular calculi with that of regular categories. In order to effect such a comparison we must first define suitable notions of 1- and 2-morphisms of regular calculi.

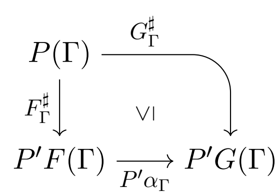
Definition 4.21 (Morphism of regular calculi)**.** Let $(\mathbb{C}_P, P)$ and $(\mathbb{C}_{P'}, P')$ be regular calculi. A **morphism of regular calculi** from $(\mathbb{C}_P, P)$ to $(\mathbb{C}_{P'}, P')$ is a pair $(F, F^\sharp)$ where $(F, \varphi)\colon \mathbb{C}_P \to \mathbb{C}_{P'}$ is a strong monoidal po-functor preserving the supply and $F^\sharp$ is a monoidal 2-natural transformation (Definition 2.9) as follows.



A 2**-morphism of regular calculi** from $(F, F^\sharp)$ to $(G, G^\sharp)$ is the data of a monoidal left adjoint oplax-natural transformation (Definitions 2.5 and 2.9) $\alpha\colon F \Rightarrow G$ which satisfies the property $F^\sharp \mathbin{⨾} P'\alpha \leq G^\sharp$ as explicated below.



i.e., for each $\Gamma \in \mathbb{C}_P$, an inequality in $\mathbb{P}\mathsf{oset}\big(P(\Gamma), P'G(\Gamma)\big)$:



(17)

Remark 4.22. The reader would of course be justified in wondering whether these particular choices of 1- and 2-morphisms preserves all of the relevant structure at hand. We provide some discussion on this topic in Section 5.4, and for greater detail still direct the reader to the companion paper [cFS21] in which we show that these morphisms support the construction of a pseudo-reflection of regular categories into regular calculi.

It is a straightforward if lengthy task to verify that morphisms of regular calculi obey strict composition laws, and moreover that 2-morphisms may be composed along shared morphism boundaries as well as whiskered with morphisms in a functorial manner. With these facts, we are justified in making the following definition.

Definition 4.23 ($\mathcal{R}$gCalc)**.** The 2**-category of regular calculi**, denoted $\mathcal{R}$gCalc, has as objects regular calculi, and as 1- and 2-morphisms the 1- and 2-morphisms of regular calculi of Definition 4.21.

Example 4.24. Recall Example 4.13, all regular calculi whose po-category of contexts is terminal are $\wedge$-semilattices. From this we may see that the full subcategory of $\mathcal{R}$gCalc spanned by regular calculi whose context category is terminal is isomorphic to the full subcategory of $\mathbb{P}$oset spanned by the $\wedge$-semilattices.

5 Graphical regular logic

In this section we finally develop our graphical formalism for regular logic by defining the notion of graphical term, showing how these graphical terms represent predicates in contexts, and explaining how to reason with them. We sketch how the collection of graphical terms, together with our graphical reasoning, allows us to form the "syntactic po-category" of a regular calculus in a 2-functorial fashion. In the companion paper [cFS21], we make extensive use of this graphical regular logic to prove the that regular categories are "pseudo-reflective" in regular calculi by means of our syntactic po-category construction.

5.1 Graphical terms

Given a regular calculus $(\mathbb{C}_P, P)$, graphical terms provide representations of its predicates, i.e. the elements in $P(\Gamma)$ for various contexts $\Gamma \in \mathbb{C}_P$.

We invite the reader to recall Definition 3.42, our definition of wiring diagrams in a po-category supplying $\mathbb{W}$, as well as Notation 3.43, our graphical notation therefor, before considering this next definition and its accompanying notation.

Definition 5.1. (Graphical term) Let $(\mathbb{C}_P, P)$ be a regular calculus. A **graphical term** is the data of

1. a wiring diagram $(k, \{\Gamma_i = (n_i, \tau_i)\}_{i \in \underline{k}}, \omega\colon \Gamma_1 \otimes \cdots \otimes \Gamma_k \to \Gamma_{\text{out}})$ in $\mathbb{C}_P$,
2. for each $i \in \underline{k}$, a predicate $\theta_i \in P(\Gamma_i)$ in context $\Gamma_i = \bigotimes_{j \in \underline{n_i}} \tau_i(j)$.

We will choose to suppress the details of the wiring diagram and notate such a graphical term by $(\theta_1, \ldots, \theta_k; \omega)$. If $k = 0$ then a graphical term $(;\omega)$ is simply a morphism $\omega\colon I \to \Gamma_{\text{out}}$.

We say that the graphical term $t = (\theta_1, \ldots, \theta_k; \omega)$ **represents** the predicate

$$[\![t]\!] := P(\omega)(\theta_1 \boxplus^k \cdots \boxplus^k \theta_k) \in P(\Gamma_{\text{out}}) \tag{18}$$

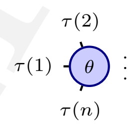
where $\boxplus^k$ is the k-ary laxator $\boxplus^k \colon \prod_{i \in \underline{k}} P(\Gamma_i) \to P(\bigotimes_{\underline{k}} \Gamma_i)$. In particular, if $k = 0$ then a graphical term $(;\omega)$ represents the predicate $P(\omega)(\texttt{true})$. We extend the equality and implication of the poset $P(\Gamma_{\text{out}})$ to graphical terms t, t' via $[\![-]\!]$ in the following sense: we say that t **implies** t' and that t **equals** t' when $[\![t]\!] \vdash [\![t']\!]$ and $[\![t]\!] = [\![t']\!]$ respectively. In a slight abuse of notation we will write $t \vdash t'$ and $t = t'$ for implication and equality in this sense.[2]

Example 5.2. When $k = 1$ and $\omega = \mathrm{id}$ is the identity, then $[\![(-;\Gamma)]\!] \colon P(\Gamma) \to P(\Gamma)$ is also the identity. More generally for any k, when $\omega = \mathrm{id}$, the map

$$[\![(-, \ldots, -; \textstyle\bigotimes_{i \in \underline{k}} \Gamma_i)]\!] \colon \prod_{i \in \underline{k}} P(\Gamma_i) \to P(\textstyle\bigotimes_{i \in \underline{k}} \Gamma_i)$$

is $\boxplus^k$, the k-ary laxator. We shall see other special cases in Proposition 5.12.

Notation 5.3 (Graphical terms)**.** We draw a graphical term $(\theta_1, \ldots, \theta_k; \omega)$ by drawing the morphism ω as in Notation 3.43 and annotating the i^{th} inner shell with its corresponding predicate θ_i. In the case that ω is the identity morphism, we may simply draw the contexts annotated by their predicates. For instance,

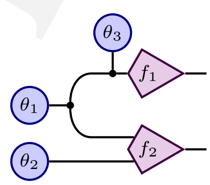
$[\![(\theta; \mathrm{id}_{\bigotimes \tau(i)})]\!]$ is represented by 

.

Example 5.4. Let $(\mathbb{C}_P, P)$ be a regular calculus, and let $\theta_i \in P(\Gamma_i)$ for $i \in \{1, 2, 3\}$ be predicates, let $f_1 \colon \Gamma_1 \to \Gamma_1'$ and $f_2 \colon \Gamma_1 \otimes \Gamma_2 \to \Gamma_2'$ be morphisms of $\mathbb{C}_P$. Let us write $\sigma \colon \Gamma_1 \otimes \Gamma_1^{\otimes 2} \otimes \Gamma_2 \to \Gamma_1^{\otimes 2} \otimes (\Gamma_1 \otimes \Gamma_2)$ for the appropriate symmetry of $\mathbb{C}_P$, then the predicate

$$[\![\big(\theta_1, \theta_2, \theta_3; (\Gamma_1 \otimes \delta_{\Gamma_1} \otimes \Gamma_2) \mathbin{⨟} \sigma \mathbin{⨟} ((\mu_{\Gamma_1} \mathbin{⨟} f_1) \otimes f_2)\big)]\!]$$

is represented by the graphical term



.

Example 5.5. We saw in Example 4.13 that right ajax po-functors $1 \to \mathbb{P}\mathrm{oset}$ are $\wedge$-semilattices. The corresponding diagrammatic language has no wires, since 1 comprises only the monoidal unit. The semantics of an arbitrary graphical term $(\theta_1, \ldots, \theta_k; \mathrm{id})$ is simply the meet $\theta_1 \wedge \cdots \wedge \theta_k$.

[2] In this sense graphical terms inherit a pre-order as well as an equivalence relation, relative to which anti-symmetry of the pre-order holds.

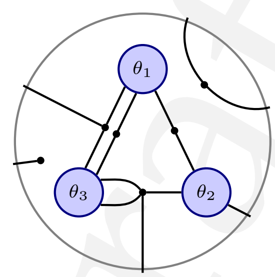
Remark 5.6. Graphical terms are an alternate syntax for regular logic. Fix a regular calculus $(\mathbb{C}_P, P)$, a context $\Gamma \in \mathbb{C}_P$ and suppose $\omega\colon \underline{n}_1 + \cdots + \underline{n}_k \to \underline{n}_{\text{out}}$ is a morphism of $\mathbb{W}$, that is, by Remark 3.7, canonically a cospan of the form

$$\underline{n}_1 + \cdots + \underline{n}_k \xrightarrow{[\omega_1,\dots,\omega_k]} \underline{n}_\omega \xleftarrow{\omega_{\text{out}}} \underline{n}_{\text{out}} \,.$$

By the supply of $\mathbb{W}$ this induces a morphism $\omega\colon \bigotimes \Gamma^{\otimes n_i} \cong \Gamma^{\otimes(\sum n_i)} \to \Gamma^{\otimes n_{\text{out}}}$. While we will not dwell on the translation, a graphical term $(\theta_1, \dots, \theta_k; \omega)$ represents the following regular formula in free variables $x_{(\text{out},1)}, \dots, x_{(\text{out},n_{\text{out}})}$.

$$\exists_{\substack{i\in\{1,\dots,k,\omega\}\\ j\in\underline{n}_i}} x_{(i,j)} \left[\bigwedge_{i'\in\{1,\dots k\}} \theta_{i'}(x_{(i',1)}, \dots, x_{(i',n_{i'})}) \quad\wedge \bigwedge_{\substack{i'\in\{1,\dots,k,\text{out}\}\\ j'\in\underline{n}_{i'}}} \left(x_{(i',j')} = x_{(\omega,\omega_{i'}(j'))}\right)\right]$$

For example, given the supplied morphism $\omega\colon \Gamma^{\otimes 3} \otimes \Gamma^{\otimes 3} \otimes \Gamma^{\otimes 4} \to \Gamma^{\otimes 6}$ of Example 3.16 and predicates $\theta_1 \in P(\Gamma^{\otimes 3})$, $\theta_2 \in P(\Gamma^{\otimes 2})$, and $\theta_3 \in P(\Gamma^{\otimes 4})$, the graphical term

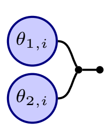


would represent the following formula after simplification.

$$\psi(y, z, z', x, x', z'') = \exists\, \tilde{x}, \tilde{y}, \left[\theta_1(\tilde{x}, \tilde{y}, y) \wedge \theta_2(x', \tilde{x}, x) \wedge \theta_3(y, \tilde{y}, x', x') \wedge (z = z')\right]$$

Before we address the *calculus* portion of our graphical notation for regular calculi, let us turn our attention to a final class of examples of our graphical notation: bare regular calculi.

Example 5.7. Recall that a regular calculus $(\mathbb{C}_P, P)$ is bare if its category of contexts $\mathbb{C}_P$ is of the form $\mathbb{C}_P = \bigsqcup_J \mathbb{W}$. Given our notation for wiring diagrams in $\mathbb{W}$, Section 3.2, we see that a graphical term $(\theta_{1,i_1}, \dots \theta_{k,i_k}; \omega)$ in $(\bigsqcup_J \mathbb{W}, P)$ is precisely a collection of wiring diagrams for $\mathbb{W}$ each of whose shells have been annotated by predicates $\theta_{j,i_j} \in P(\underline{n}_{j,i_j})$ where $j, i_j \in J$. A somewhat typical example might therefore be



where $\theta_{1,j}, \theta_{2,j} \in P(\underline{1}_j)$ for $j \in J$ and $\omega = \mu \,⨾\, \epsilon\colon \underline{1}_j + \underline{1}_j \to \underline{0}_j$.

As there are no labelled morphisms decorating its graphical terms and instead only wires, the regular calculus $(\bigsqcup_J \mathbb{W}, P)$ is in this visual sense considered *bare*.

5.2 Reasoning with graphical terms

Now that we have understood the graphical notation, it is time to attend to the calculus of manipulations it supports. Let $(\mathbb{C}_P, P)$ be a regular calculus. The following basic rules for reasoning with graphical terms express the 2-functoriality and monoidality of the po-functor $P\colon \mathbb{C}_P \to \mathbb{P}\text{oset}$.

Proposition 5.8. *Let $(\theta_1, \ldots, \theta_k; \omega)$ be a graphical term, where $\theta_i \in P(\Gamma_i)$.*

i. (Monotonicity) Suppose $\theta_i \vdash \theta_i'$ for some i. Then

$$[\![(\theta_1, \ldots, \theta_i, \ldots, \theta_k; \omega)]\!] \vdash [\![(\theta_1, \ldots, \theta_i', \ldots, \theta_k; \omega)]\!].$$

ii. (Breaking) Suppose $\omega \leq \omega'$ in $\mathbb{C}_P$. Then

$$[\![(\theta_1, \ldots, \theta_k; \omega)]\!] \vdash [\![(\theta_1, \ldots, \theta_k; \omega')]\!].$$

iii. (Nesting) Suppose $\theta_i = [\![(\theta_1', \ldots, \theta_\ell'; \omega')]\!]$ for some $1 \leq i \leq k$. Then

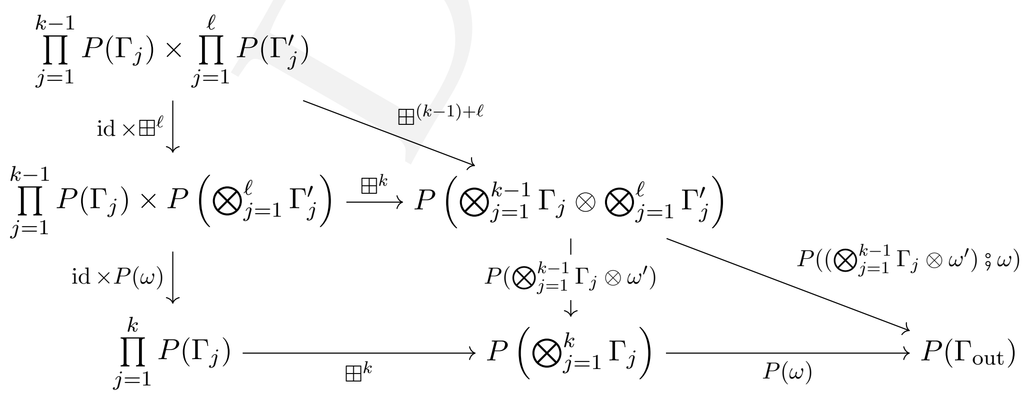
$$\begin{aligned}&[\![(\theta_1, \ldots, \theta_{i-1}, [\![(\theta_1', \ldots, \theta_\ell'; \omega')]\!], \theta_{i+1}, \ldots, \theta_k; \omega)]\!] = [\![(\theta_1, \ldots, \theta_i \ldots, \theta_k; \omega)]\!] \\ &= [\![(\theta_1, \ldots, \theta_{i-1}, \theta_1', \ldots, \theta_\ell', \theta_{i+1}, \ldots, \theta_k; (\textstyle\bigotimes_{1\leq j<i} \Gamma_j \otimes \omega' \otimes \bigotimes_{i<j\leq k} \Gamma_j) \,;\, \omega)]\!]\end{aligned}$$

Proof. By examining $[\![-]\!]$ of Definition 5.1 we may reason as below.

(i) This claim follows from the monotonicity of the map $\boxplus^k \,;\, P(\omega)$.

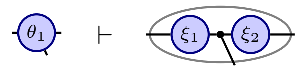
(ii) This claim follows from the 2-functoriality of P.

(iii) This claim follows from the monoidality and 1-functoriality of P. By using the symmetry of $\mathbb{C}_P$, without loss of generality we may assume that $i = k$. In this case, to prove the desired equality it is sufficient to demonstrate the commutativity of the following diagram.

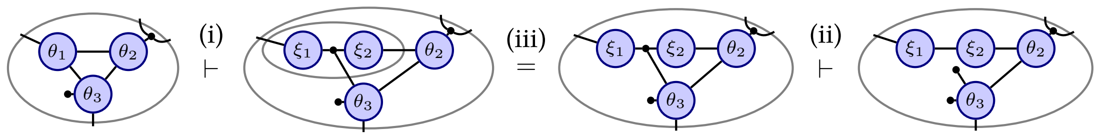


In the above diagram, the upper triangle commutes by coherence laws for $\boxplus$, the square commutes by naturality of $\boxplus$, and the right hand triangle commutes by functoriality of P. ☐

Example 5.9. Proposition 5.8 is perhaps more quickly grasped through a graphical example of these facts in action. Suppose we have the entailment



Then using monotonicity, nesting, and then breaking we may deduce the entailment



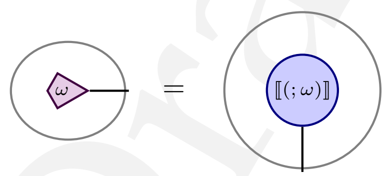
We'll see many further examples in [cFS21], where we prove that we can construct a regular category from a regular calculus.

The nesting rule of Proposition 5.8 (iii) has two particularly important cases.

Example 5.10 (Wiring diagrams as predicates). Let $\omega\colon I \to \Gamma$ be a morphism in $\mathbb{C}_P$. Observe that we have the equalities

$$\begin{aligned} [\![(;\omega)]\!] &= \big(1 \xrightarrow{\ \texttt{true}\ } P(I) \xrightarrow{\ P(\omega)\ } P(\Gamma)\big) \\ &= \big(1 \xrightarrow{\ \texttt{true}\,⨾P(\omega)\ } P(\Gamma) \xrightarrow{\ P(\Gamma)\ } P(\Gamma)\big) = [\![([\![(;w)]\!];\Gamma)]\!] \quad , \end{aligned}$$

so that we are justified in equating the following two graphical terms



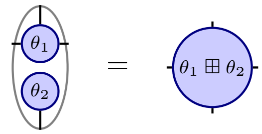
Moreover, every regular calculus has a rich stock of such morphisms $I \to \Gamma$. In Lemma 3.40 we exhibited an isomorphism $\mathbb{C}_P(\bigotimes\Gamma_i, \Gamma_{\text{out}}) \cong \mathbb{C}_P(I, \bigotimes\Gamma_i \otimes \Gamma_{\text{out}})$ mediated by taking the name $\omega \mapsto \omega^{\ulcorner}$ of a morphism. In this way we may view arbitrary wiring diagrams $\omega\colon \bigotimes\Gamma_i \to \Gamma_{\text{out}}$ in $\mathbb{C}_P$ as graphical terms $[\![([\![(;\omega^{\ulcorner})]\!];\bigotimes\Gamma_i \otimes \Gamma_{\text{out}})]\!]$ of the above-right form.

Note, however, that in general the above merely constitutes a map of wiring diagrams into predicates which preserves representation. It is not necessarily the case that all predicates $\theta \in P(\Gamma)$ may be realised as $[\![([\![(;\omega)]\!];\Gamma)]\!]$ for some wiring diagram ω. Nevertheless, in Section 6.4 we will see that there is a large class of regular calculi in which wiring diagrams and predicates do coincide.

Example 5.11 (Exterior conjunction). Let Γ_1 and Γ_2 be contexts, and let $\theta_1 \in P(\Gamma_1)$ and $\theta_2 \in P(\Gamma_2)$ be predicates. Observe that we have the equalities

$$\theta_1 \boxplus \theta_2 = [\![(\theta_1 \boxplus \theta_2; \Gamma_1 \otimes \Gamma_2)]\!] = [\![(\theta_1, \theta_2; \Gamma_1 \otimes \Gamma_2)]\!]$$

of elements of $P(\Gamma_1 \otimes \Gamma_2)$ so that we are justified in equating, for example, the following two graphical terms.

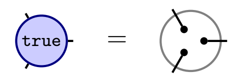


Under the interpretation of graphical terms as formulae in regular logic suggested by Remark 5.6, this process of vertical merging of graphical terms corresponds to the logical conjunction of the formulae they represent.

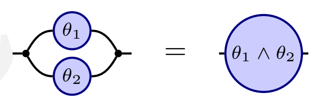
In Proposition 4.14 we saw how a regular calculus endows each poset $P(\Gamma)$ with the structure of a meet-semilattice. As we will now see, this structure permits an intuitive graphical interpretation. In the following proposition, the graphical terms on right are illustrative examples of the equalities stated on the left.

Proposition 5.12. *For all contexts $\Gamma \in \mathrm{Ob}\,\mathbb{C}_P$ and predicates $\theta_1, \theta_2 \in P(\Gamma)$, we have*

i. (True is removable) $[\![(\mathtt{true}_\Gamma;\Gamma)]\!] = [\![(;\eta_\Gamma)]\!]$,



ii. (Meets-are-merges) $[\![(\theta_1 \wedge \theta_2;\Gamma)]\!] = [\![(\theta_1,\theta_2;\mu_\Gamma)]\!]$.



Proof. These equations are simply the definitions of $\mathtt{true}$ and meet; see (16) and (18). $\square$

Example 5.13 (Discarding). Note that Proposition 5.12 (i) and the monotonicity of diagrams (Proposition 5.8 (i)) further imply that for all $\theta \in P(\Gamma)$ we have $\theta \vdash [\![(;\eta_\Gamma)]\!]$:

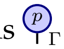
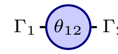
5.3 The syntactic po-category of a regular calculi: a sketch

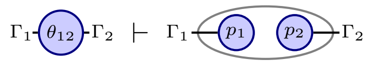
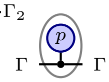
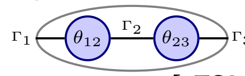
As we have seen, graphical terms provide an effective way to reason in regular calculi. It is thus of interest to consider forming, from a given regular calculus $(\mathbb{C}_P, P)$, a po-category $\mathbb{S}\mathsf{yn}(\mathbb{C}_P, P)$ whose *objects* are the graphical terms of $(\mathbb{C}_P, P)$. In this way we may study the collection of representations for predicates at once – that is, we may study the *syntax* of the regular calculus.

In what follows we will sketch the "syntactic po-category" construction, but we will choose here to defer the full details to the companion paper. Consider then the following collections of data which together form the objects and morphisms of the **syntactic po-category** $\mathbb{S}\mathsf{yn}(\mathbb{C}_P, P)$ of the regular calculus $(\mathbb{C}_P, P)$.

$$\begin{cases} \mathrm{Ob}\,\mathbb{S}\mathsf{yn}(\mathbb{C}_P, P) & := \{(\Gamma, p) \mid \Gamma \in \mathrm{Ob}\,\mathbb{C}_P, p \in P(\Gamma)\} \\ \mathbb{S}\mathsf{yn}(\mathbb{C}_P, P)\big((\Gamma_1, p_1), (\Gamma_2, p_2)\big) & := \{\theta_{12} \in P(\Gamma_1 \otimes \Gamma_2) \mid \theta_{12} \leq p_1 \boxplus p_2\} \end{cases} \tag{19}$$

Of course to claim that these data form a po-category we must provide various additional structures and prove properties thereof. While it is possible to continue our sketch in the language of supplied morphisms and right ajax po-functors – that is, semantically – we will instead make use of the tools of graphical regular logic. Thus, in pictures, the to-be po-category $\mathbb{S}\mathsf{yn}(\mathbb{C}_P, P)$ has:

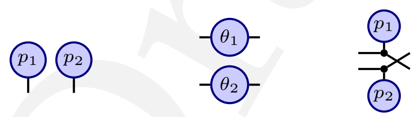
- objects (Γ, p) represented by graphical terms 
- morphisms $\theta_{12}\colon (\Gamma_1, p_1) \to (\Gamma_2, p_2)$ represented by graphical terms 

  together with an entailment 
- the identity on (Γ, p) represented by the graphical term 
- the composite $\theta_{12} \mathbin{⨾} \theta_{23}$ represented by the graphical term 

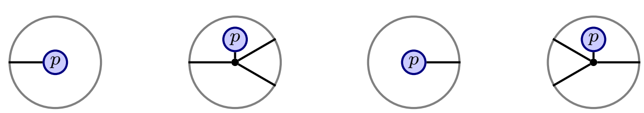
With the induced poset structure on the homs, in the companion paper [cFS21] we prove that the collections $\mathbb{S}\mathsf{yn}(\mathbb{C}_P, P)$ with the composition and identities above indeed form a po-category.

Example 5.14. In Example 4.13 we established that $\wedge$-semilattices L are equivalently regular calculi $(1, L)$. By unwinding the syntactic po-category construction for such a regular calculus we see that $\mathbb{S}\mathsf{yn}(1, L)$ is the po-category whose objects are the elements of L, whose hom posets $\mathbb{S}\mathsf{yn}(1, L)(l, l')$ are the down-sets $\downarrow\{l \wedge l'\}$, whose composition is meet, and whose identities are given by the top element of each hom poset.

The syntactic po-category moreover inherits from $\mathbb{C}_P$ a canonical symmetric monoidal structure where the monoidal product of objects, the monoidal product of morphisms, and the braiding correspond respectively to the following graphical terms – for details see [cFS21].



With this symmetric monoidal structure we may induce, from the supply of $\mathbb{W}$ in $\mathbb{C}_P$, a supply of $\mathbb{W}$ in $\mathbb{S}\mathsf{yn}(\mathbb{C}_P, P)$. This work appears as [cFS21], but we summarise the results here. Recall that ϵ, δ, η, and μ are the generating morphisms of $\mathbb{W}$; see (3). For an object $(\Gamma, p) \in \operatorname{Ob} \mathbb{S}\mathsf{yn}(\mathbb{C}_P, P)$, the supplied morphisms corresponding to these generators are the following graphical terms.



$\epsilon_{(\Gamma,p)}$ $\delta_{(\Gamma,p)}$ $\eta_{(\Gamma,p)}$ $\mu_{(\Gamma,p)}$

In fact there is even more coherent structure present, in the terms of Section 6.2 $\mathbb{S}\mathsf{yn}(\mathbb{C}_P, P)$ is a "relational" po-category, but we will delay this statement and its implications to that section.

Remark 5.15. Observe that $\mathbb{P}\mathsf{oset}$ is a sub-2-category of $\mathfrak{C}\mathsf{at}$. Instead of our bespoke construction of the syntactic po-category above, it is tempting to consider some appropriate po-categorical variant of a monoidal Grothendieck construction – perhaps as developed in [MV18] or [Buc13]. Indeed, at the level of objects it would seem that there is a co-incidence between $\mathbb{S}\mathsf{yn}(\mathbb{C}_P, P)$ and the total space of a Grothendieck-type construction $\int(\mathbb{C}_P, P)$.

However, it presently appears to the authors that any so-attempted recasting of $\mathbb{S}\mathsf{yn}$ is doomed to failure. In a putative Grothendieck construction, consider the pair of objects $(I, \mathtt{true})$ and (Γ, p). In order for the total space to supply $\mathbb{W}$, we would require the presence of morphisms $\widehat{\epsilon}\colon (\Gamma, p) \to (I, \mathtt{true})$ and $\widehat{\eta}\colon (I, \mathtt{true}) \to (\Gamma, p)$. However, in general we have only $P(\epsilon)(p) \vdash \mathtt{true}$ and $p \vdash P(\eta)(\mathtt{true})$, and thus there appears to be no uniform way to select the direction of the inequalities for morphisms in $\int(\mathbb{C}_P, P)$.

In this way, some form of 'symmetrisation' of domain and codomain becomes necessary, considerations of which result in our $\mathbb{S}\mathsf{yn}(\mathbb{C}_P, P)$.

Although many details of the syntactic po-category construction do work, and we are able to prove the central result Theorem 7.10 and several interesting corollaries, nevertheless forming the syntactic po-category in this manner can lose some information. Despite the fact that in later sections and the companion paper we shall realise $\mathbb{S}\mathsf{yn}$ as a highly-structured 2-functor as in the above theorem, it fails to mediate any form of equivalence. That is, the following counter-example stands to establish that a regular calculus is *more* than the data of its graphical terms.

Counter-example 5.16 ($\mathbb{S}\mathsf{yn}$ identifies distinct regular calculi). Recall Example 4.13, that is, that $\wedge$-semilattices L are equivalently regular calculi $L\colon 1 \to \mathbb{P}\mathsf{oset}$. Consider then that we may form the degenerate regular calculus $(\mathbb{C}, \mathbb{C} \xrightarrow{!} 1 \xrightarrow{L} \mathbb{P}\mathsf{oset})$ for any $\mathbb{C}$ supplying $\mathbb{W}$. The syntactic po-categories of these degenerate regular calculi are equally degenerate: in $\mathbb{S}\mathsf{yn}(\mathbb{C}, \mathbb{C} \xrightarrow{!} 1 \xrightarrow{L} \mathbb{P}\mathsf{oset})$ there is a (natural) isomorphism $(c, p) \cong (c', p)$ for all elements $p \in L$ and objects $c, c' \in \operatorname{Ob}\mathbb{C}$, viz, p itself.

It may be checked that given $\mathbb{C}$ and $\mathbb{D}$ inequivalent symmetric monoidal po-categories supplying $\mathbb{W}$, the regular calculi $(\mathbb{C}, !_{\mathbb{C}} \mathbin{⨾} L)$ and $(\mathbb{C}, !_{\mathbb{C}} \mathbin{⨾} L)$ are inequivalent. However, the po-functor $\mathbb{S}\mathsf{yn}(\mathbb{C}, !_{\mathbb{C}} \mathbin{⨾} L) \to \mathbb{S}\mathsf{yn}(\mathbb{D}, !_{\mathbb{D}} \mathbin{⨾} L)$ which sends $(c, p) \mapsto (I^{\mathbb{D}}, p)$ and $\theta \mapsto \theta$ mediates an equivalence (with evident inverse) of symmetric monoidal po-categories supplying $\mathbb{W}$. Thus $\mathbb{S}\mathsf{yn}$ cannot mediate a 2-dimensional equivalence of 2-categories.

Remark 5.17. In fact this failure is part of a more general class. In Proposition 6.18 below we record a result of the companion paper: whenever there is a morphism of regular calculi $(F, F^\sharp)\colon \mathbb{C}_P \to \mathbb{C}_Q$ such that F is split essentially surjective (Definition 2.7) and $F^\sharp$ is an isomorphism then $\mathbb{S}\mathsf{yn}(F, F^\sharp)\colon \mathbb{S}\mathsf{yn}\,\mathbb{C}_P \to \mathbb{S}\mathsf{yn}\,\mathbb{C}_Q$ mediates an equivalence.

Observe then that for any po-category $\mathbb{C}$ supplying $\mathbb{W}$, the canonical morphism $(!_{\mathbb{C}}, \mathrm{id}_L)\colon (\mathbb{C}, !_{\mathbb{C}} \mathbin{⨾} L) \to (1, L)$ of regular calculi satisfies these conditions. Thus we may conclude that $\mathbb{S}\mathsf{yn}(\mathbb{C}, !_{\mathbb{C}} \mathbin{⨾} L) \simeq \mathbb{S}\mathsf{yn}(1, L)$, which implies the above counter-example.

5.4 Morphisms of regular calculi & graphical terms

Our notions of 1- and 2-morphisms of regular calculi, Definition 4.21, interact well with graphical terms and indeed preserve all of the desired structure. We have just seen the sense in which graphical terms in a regular calculus are meaningfully the objects of a syntactic po-category, so we now elucidate the manner in which morphisms of regular calculi $(\mathbb{C}_P, P) \to (\mathbb{C}_Q, Q)$ act on graphical terms.

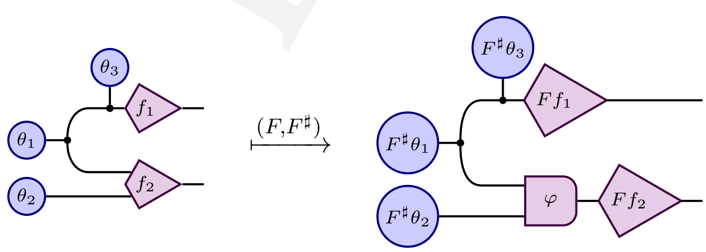
Given a morphism $(F, F^\sharp)\colon (\mathbb{C}_P, P) \to (\mathbb{C}_Q, Q)$ of regular calculi and a collection of contexts $\{\Gamma_i\}_{i\in\{1,\dots,k,\mathrm{out}\}}$ in $\mathbb{C}_P$, the monoidal 2-naturality of $F^\sharp$ renders commutative the following diagram.

$$\begin{array}{ccccccc}
\prod P(\Gamma_i) & & \xrightarrow{\qquad\boxplus\qquad} & & P(\bigotimes\Gamma_i) & \xrightarrow{P(\omega)} & P(\Gamma_{\mathrm{out}}) \\
\big\downarrow{\scriptstyle \prod F^\sharp_{\Gamma_i}} & & & & & & \big\downarrow{\scriptstyle F^\sharp_{\Gamma_{\mathrm{out}}}} \\
\prod P'F(\Gamma_i) & \xrightarrow[\boxplus']{} & P'(\bigotimes' F(\Gamma_i)) & \xrightarrow[P'\varphi]{} & P'F(\bigotimes\Gamma_i) & \xrightarrow[P'F(\omega)]{} & P'F(\Gamma_{\mathrm{out}})
\end{array}$$

Thus, given a graphical term $(\theta_1, \dots, \theta_k; \omega)$ of $(\mathbb{C}_P, P)$ where $\theta_i \in P(\Gamma_i)$, we see that we obtain the graphical term $(F^\sharp_{\Gamma_1}(\theta_1), \dots, F^\sharp_{\Gamma_k}(\theta_k); \varphi \mathbin{⨟} F(\omega))$ of $(\mathbb{C}_Q, Q)$ with the property

$$[\![(F^\sharp_{\Gamma_1}(\theta_1), \dots, F^\sharp_{\Gamma_k}(\theta_k); \varphi \mathbin{⨟} F(\omega))]\!] = F^\sharp_{\Gamma_{\mathrm{out}}}([\![(\theta_1, \dots, \theta_k; \omega)]\!])\,.$$

The fact that φ is the strongator of the supply-preserving strong symmetric monoidal po-functor F affords us an easy graphical understanding of this action. First we replace all the predicates $\theta_i \in P(\Gamma_i)$ in shells with the predicates $F^\sharp_{\Gamma_i}(\theta_i)$, and then when ω is composed of tensors of morphisms $\omega = \bigotimes \omega_i$, we may "pull φ through the tensors" in $F(\omega)$ and preserve wiring as we go. These principles are illustrated by the following example.



This description suggests that morphisms of regular calculi preserve the connectivity, wiring, and compositionality of graphical terms, and so all the structure present in our syntactic po-categories. Indeed, as we prove in the companion paper [cFS21], such a morphism $(F, F^\sharp)\colon (\mathbb{C}_P, P) \to (\mathbb{C}_Q, Q)$ of regular calculi induces a symmetric monoidal supply preserving po-functor $\mathbb{S}\mathsf{yn}(F, F^\sharp)\colon \mathbb{S}\mathsf{yn}(\mathbb{C}_P, P) \to \mathbb{S}\mathsf{yn}(\mathbb{C}_Q, Q)$. In this fashion, we may prove that $\mathbb{S}\mathsf{yn}$ forms a 2-functor

$$\mathbb{S}\mathsf{yn}\colon\ \mathcal{R}\mathsf{gCalc} \to \{\text{symmetric monoidal po-categories supplying } \mathbb{W}\}\ .$$

However, as we stated in the previous section, $\mathbb{S}\mathsf{yn}$ is in fact valued in "relational po-categories" (see Section 6.2 and Proposition 6.18), and it is at that level that we shall seek to establish its 2-functoriality.

6 Relational po-categories

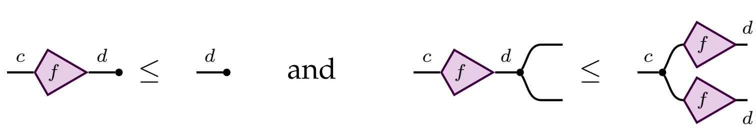
The goal of this section is at first to axiomatise those po-categories whose objects, morphisms, and 2-morphisms arise as the objects, relations, and inclusions of relations in a regular category. We call these po-categories *relational* and introduce this notion and the 2-category $\mathcal{R}\mathsf{lPoCat}$ of such in Section 6.2. As we shall see there, relational po-categories admit tidy definition in terms of the notions of supply of Section 3.3, the po-category for wiring $\mathbb{W}$, and an extra piece of structure due to Freyd and Scedrov [FS90]. Moreover, as we note in Theorem 6.19, the 2-category of relational po-categories is appropriately equivalent to the 2-category of regular categories. The fundamental impetus for so characterising regular categories is the construction of the 2-functor $\mathbb{P}\mathsf{rd}$ of Section 6.3, from relational po-categories to regular calculi. In our main theorem, Theorem 7.10, we show that this 2-functor $\mathbb{P}\mathsf{rd}$ is 2-dimensionally fully-faithful and is a 2-dimensional right adjoint, and thereby affirm that regular calculi and graphical logic may be leveraged as tools for studying relational po-categories and so regular categories.

6.1 The 2-category $\mathcal{P}\mathsf{rlPoCat}$ of prerelational po-categories

The first step towards characterising relations in regular categories is the notion of a prerelational category. Whereas relational po-categories play host to a full regular structure, prerelational po-categories support only the underlying cartesian structure – see [FS19b, Section 8.1] for details.

Definition 6.1 (Prerelational po-category)**.** A symmetric monoidal po-category $(\mathbb{R}, I, \otimes)$ is **prerelational** if it supplies wiring $\mathbb{W}$, such that the induced supply of cocommutative comonoids (Example 3.32) is lax homomorphic.

Remark 6.2. The lax homomorphicity condition requires that for every $f\colon c \to d$ in $\mathbb{R}$, both $f \mathbin{⨟} \epsilon_d \leq \epsilon_c$ and $f \mathbin{⨟} \delta_d \leq \delta_c \mathbin{⨟} (f \otimes f)$. In graphical notation:



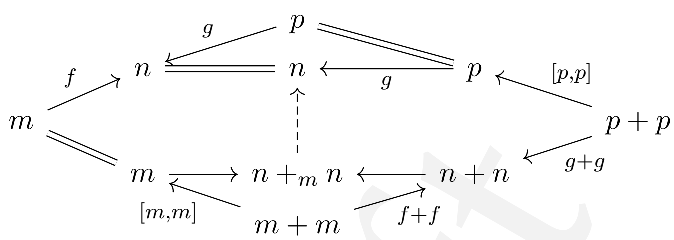
(20)

As follows from the mate calculus, or indeed through pleasing graphical manipulations, we may re-characterise prerelational po-categories in terms of the induced supply of commutative monoids instead.

Lemma 6.3. *Suppose $\mathbb{C}$ supplies wiring $\mathbb{W}$. Then the induced supply of comonoids is lax homomorphic iff the supply of monoids is oplax homomorphic.*

Proof. Note that given l left adjoint to r, we always have that $a \leq (l \mathbin{;} b)$ iff $(r \mathbin{;} a) \leq b$ and that $(a \mathbin{;} l) \leq b$ iff $a \leq (b \mathbin{;} r)$. The result now follows from Proposition 3.9. $\square$

Example 6.4. The po-category $\mathbb{W}$ is prerelational. It supplies $\mathbb{W}$ by Proposition 3.30. To see that the supply of comonoids is lax homomorphic, take a cospan $m \xrightarrow{f} n \xleftarrow{g} p$. In the case of the diagonal, the necessary composites are computed as pushouts (e.g. $n +_m n$), and the inequality from (20) is indicated by the dotted arrow, in the diagram



The case of co-units is similar.

The po-category of finite sets and corelations is also prerelational; see Example 3.34.

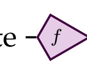
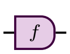
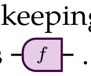
Example 6.5. More generally, given any category $\mathcal{C}$ with finite limits, the poset reflection of $\mathbb{S}\mathsf{pan}(\mathcal{C})$ is prerelational. If $\mathcal{C}$ is regular, then $\mathbb{R}\mathsf{el}(\mathcal{C})$ is prerelational.

A class of examples for which we will have use later is given by the following lemma.

Lemma 6.6. *Let I be a set and $\mathbb{R}$ a prerelational po-category supplying $\mathbb{W}$. Then with induced supply of $\mathbb{W}$ in $\bigsqcup_J \mathbb{R}$ of Lemma 3.28, the symmetric monoidal po-category $\bigsqcup_J \mathbb{R}$ is prerelational.*

Proof. Direct computation. $\square$

By [FS19b, Corollary 6.2], if $f\colon r \to s$ is a left adjoint in a prerelational po-category $(\mathbb{R}, I, \otimes)$ then its right adjoint is its transpose $f^\dagger\colon r \to s$. With this fact we are justified in introducing the following notation.

Notation 6.7 (Left adjoints)**.** We shall denote  by  when $f\colon r \to s$ is known to be a left adjoint. In keeping with Notation 3.38, we shall denote its right adjoint and transpose $f^\dagger\colon s \to r$ as  .

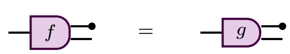
Remark 6.8. Left adjoints in a prerelational po-category are profitably understood as the true morphisms of the cartesian 1-category whose structure has been elaborated. As such, we should expect all analogues of cartesian results for left adjoints, such as: two left adjoints $f, g\colon r \to s \otimes t$ are equal iff their "projections" under ϵ_s and ϵ_t agree; left adjoints are monoid homomorphisms strictly, for to them the structure is cartesian; a "natural transformation" between "cartesian" po-functors of prerelational po-categories is automatically "monoidal" if its components are left adjoints. Indeed we make the first and last results precise and prove them as Lemmas 6.10 and 6.12 later, while the middle appears as [FS19b, Corollary 6.2]. This serves as additional motivation to distinctly signify left adjoints graphically.

Definition 6.9. The 2**-category** $\mathcal{P}\mathsf{rlPoCat}$ **of prerelational po-categories** has as objects the prerelational po-categories, as morphisms the strong symmetric monoidal po-functors, and as 2-morphisms the left adjoint oplax-natural transformations $\alpha\colon F \Rightarrow G$; see Definition 2.5.

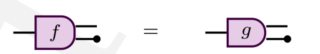
Although it is not immediate, the 1 and 2-morphisms we have chosen above are indeed appropriate for respecting *all* the structure present in prerelational po-categories. To show this we appeal to a few results of [FS19b].

Lemma 6.10 ([FS19b, Proposition 6.22])**.** *If $\mathbb{R}$ and $\mathbb{R}'$ are prerelational po-categories and $F\colon \mathbb{R} \to \mathbb{R}'$ is any strong symmetric monoidal po-functor, then F automatically preserves the supply of $\mathbb{W}$.*

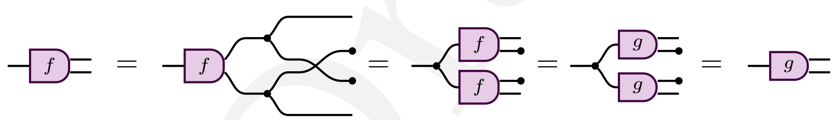
Lemma 6.11. *If $\mathbb{R}$ is prerelational, and $f, g\colon r \to s \otimes s'$ are two left adjoints, then $f = g$ iff*



and



Proof. Certainly $f = g$ implies the given condition. For the converse recall that by [FS19b, Corollary 6.2] left adjoints are comonoid homomorphisms – the inequalities of (20) are equalities –, so that we may argue as follows.



∎

Lemma 6.12. *Let $(F, \varphi), (G, \psi)\colon \mathbb{R} \to \mathbb{R}'$ be strong symmetric monoidal po-functors between prerelational po-categories, and let $\alpha\colon F \Rightarrow G$ be a left adjoint oplax-natural transformation. Then α is a monoidal oplax-natural transformation (Definition 2.9).*

Proof. We must show that the diagrams

$$\begin{array}{ccc} I' & & \\ {\scriptstyle \varphi_I}\downarrow & \searrow^{\psi_I} & \\ FI & \xrightarrow[\alpha_I]{} & GI \end{array} \qquad \text{and} \qquad \begin{array}{ccc} Fr \otimes' Fs & \xrightarrow{\alpha_r \otimes \alpha_s} & Gr \otimes' Gs \\ {\scriptstyle \varphi_{r,s}}\downarrow & & \downarrow{\scriptstyle \psi_{r,s}} \\ F(r \otimes s) & \xrightarrow[\alpha_{r \otimes s}]{} & G(r \otimes s) \end{array}$$

are strictly commutative for all objects $r, s \in \operatorname{Ob}\mathbb{R}$. Observe that all the strongators are isomorphisms and so are, in particular, left adjoints. Thus these are diagrams of left adjoints and by [FS19b, Proposition 6.5] it suffices to show in the first case that $\varphi_I \mathbin{⨟} \alpha_I \le \psi_I$, equivalently $\alpha_I \mathbin{⨟} \psi_I^{-1} \le \varphi_I^{-1}$. In the second case we will use Lemma 6.11 to show $\varphi_{r,s} \mathbin{⨟} \alpha_{r\otimes s} \mathbin{⨟} \psi^{-1} = \alpha_r \otimes \alpha_s$.

In the first case, by Lemma 6.10 above, we have the equalities $\varphi_I^{-1} = \epsilon'_{FI} \colon FI \to I$ and $\psi_I^{-1} = \epsilon'_{GI} \colon GI \to I$ as both (F, φ) and (G, ψ) preserve the supply of $\mathbb{W}$. As $\mathbb{R}'$ is prerelational, $\alpha_I \colon FI \to GI$ must be lax homomorphic and so we have

$$\alpha_I \mathbin{;} \psi_I^{-1} = \alpha_I \mathbin{;} \epsilon'_{GI} \leq \epsilon'_{FI} = \varphi_I^{-1} \ ,$$

as desired.

To show the equality of the left adjoints $\varphi_{r,s} \mathbin{;} \alpha_{r \otimes s} \mathbin{;} \psi^{-1}$ and $\alpha_r \otimes \alpha_s$ let us begin by writing $\pi'_{Gr} \coloneqq (\epsilon'_{Gr} \otimes \mathrm{id}_{Gs}) \mathbin{;} \lambda'_{Gs} \colon Gr \otimes' Gs \to Gs$ and likewise for π'_{Gs}, π'_{Fr}, and π'_{Fs}. Lemma 6.11 shows that it is enough to prove that

$$\varphi_{r,s} \mathbin{;} \alpha_{r \otimes s} \mathbin{;} \psi^{-1} \mathbin{;} \pi'_{Gr} = \alpha_r \otimes \alpha_s \mathbin{;} \pi'_{Gr} \quad \text{and} \quad \varphi_{r,s} \mathbin{;} \alpha_{r \otimes s} \mathbin{;} \psi^{-1} \mathbin{;} \pi'_{Gs} = \alpha_r \otimes \alpha_s \mathbin{;} \pi'_{Gs} \ .$$

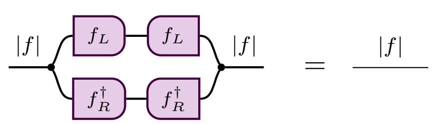
The proofs that these equalities hold involve large diagrams which make use of the commutativity of the triangle established above, [FS19b, Proposition 6.5] to see that the oplax-naturality squares for α on left adjoints are actually strict, and Lemma 6.10 for equalities like $G(\epsilon_r) \mathbin{;} \psi_I^{-1} = \epsilon'_{Gr}$. These diagrams are not especially illuminating and so we have not included them here. Nevertheless, with the aforementioned techniques, it is possible to show that both terms of the above-left claimed equality reduce to $\pi'_{Fs} \mathbin{;} \alpha_s$, while for the above-right claimed equality the reduced form is $\pi'_{Fr} \mathbin{;} \alpha_r$. □

6.2 The 2-category $\mathcal{R}\mathsf{lPoCat}$ of relational po-categories

To make a prerelational po-category relational, we need tabulations. The following definition is due to Freyd and Scedrov [FS90].

Definition 6.13 (Tabulation)**.** Suppose $\mathbb{C}$ supplies $\mathbb{W}$ and let $f \colon r \to s$ be a morphism in $\mathbb{C}$. A **tabulation** (f_R, f_L) **of** f is a factorization $r \xrightarrow{f_R} |f| \xrightarrow{f_L} s$ of f where

(i) $f_R \colon r \to |f|$ is a right adjoint in $\mathbb{C}$;
(ii) $f_L \colon |f| \to s$ is a left adjoint in $\mathbb{C}$; and
(iii) $\hat{f} \mathbin{;} \hat{f}^\dagger = \mathrm{id}_{|f|}$, where $\hat{f} \coloneqq \delta_{|f|} \mathbin{;} (f_L \otimes f_R^\dagger)$; in pictures



(21)

Definition 6.14 (Relational po-category)**.** A **relational po-category** is a prerelational po-category $\mathbb{R}$ in which additionally every morphism has a a chosen tabulation.

The 2**-category** $\mathcal{R}\mathsf{lPoCat}$ **of relational po-categories** is the 2-full sub-2-category $\mathcal{P}\mathsf{rlPoCat}$ whose objects are relational po-categories. That is, it has as objects relational po-categories, as morphisms strong symmetric monoidal po-functors, and as 2-morphisms the left adjoint oplax-natural transformations.

By Lemma 6.10 we thus know that strong symmetric monoidal po-functors between relational po-categories preserve the supply, but in fact more is true.

Lemma 6.15 ([FS19b, Proposition 6.22])**.** *If $\mathbb{R}$ and $\mathbb{R}'$ are relational po-categories and $F\colon \mathbb{R} \to \mathbb{R}'$ is any strong symmetric monoidal po-functor, then F automatically preserves the supply of $\mathbb{W}$ and tabulations.*

Remark 6.16. A prerelational po-category is exactly what Carboni and Walters called a *'bicategory of relations'* (quotation marks are an explicit part of the their terminology), and a relational po-category is exactly what they called a *functionally complete 'bicategory of relations'*. There are a few, ultimately immaterial differences in the definition; see [FS19b, Section 8.1] for details.

Example 6.17. As seen in Example 6.4, the po-category $\mathbb{W}$ is prerelational. However, it is not relational: the cospan $\underline{0} \to \underline{1} \leftarrow \underline{0}$ in $\mathbb{W}$ does not have a tabulation. On the other hand, the po-category of finite sets and co-relations is relational.

As was suggested in Section 5.3, the syntactic po-category of a regular calculus is in fact a relational po-category. Thus we have an abundant source of relational po-categories. We will merely record this fact here, and defer the details to the companion paper.

Proposition 6.18 ([cFS21, ???])**.** *The syntactic po-category construction is the on-objects component of a 2-functor $\mathbb{S}\mathsf{yn}\colon \mathcal{R}\mathsf{gCalc} \to \mathcal{R}\mathsf{lPoCat}$. Moreover, if $(F, F^\sharp)\colon \mathbb{C}_P \to \mathbb{C}_Q$ is a morphism of regular calculi such that F is split essentially surjective and $F^\sharp$ is an isomorphism then $\mathbb{S}\mathsf{yn}(F, F^\sharp)$ is an equivalence.*

Every regular category has a po-category of relations, and it is relational; this mapping extends to a 2-functor $\mathbb{R}\mathsf{el}\colon \mathcal{R}\mathsf{gCat} \to \mathcal{R}\mathsf{lPoCat}$ sending a regular category $\mathcal{R}$ to the po-category with the same objects and with hom-posets given by

$$\mathbb{R}\mathsf{el}(\mathcal{R})(r_1, r_2) := \mathsf{Sub}_{\mathcal{R}}(r_1 \times r_2). \tag{22}$$

In the other direction, the category of left adjoints in any relational po-category is regular; this also extends to a 2-functor $\mathsf{LAdj}\colon \mathcal{R}\mathsf{lPoCat} \to \mathcal{R}\mathsf{gCat}$. It turns out these functors form an equivalence of 2-categories. Indeed, this is the Carboni-Walters idea, although they did not explicitly give the full 2-categorical account. However, this equivalence was proven as the main theorem in [FS19b, Theorem 7.3].

Theorem 6.19. *The 2-functors $\mathbb{R}\mathsf{el}\colon \mathcal{R}\mathsf{gCat} \leftrightarrows \mathcal{R}\mathsf{lPoCat} : \mathsf{LAdj}$ form an equivalence of 2-categories. Their underlying 1-functors moreover form an equivalence between the underlying 1-categories.*

Example 6.20. Under the equivalence from Theorem 6.19, the relational po-category of finite sets and equivalence relations corresponds to the regular category $\mathsf{FinSet}^{\mathrm{op}}$. This is *almost* the free regular category on one object, but not quite: it is the free regular category in which every object x is inhabited (the unique map $x \to 1$ is a regular epi).

Remark 6.21. In fact, more is true of LAdj. As a consequence of the main result Theorem 7.10 and the work of companion paper [cFS21], in Corollary 7.5 below we prove that LAdj is "bi-represented" by the relational po-category, $\mathbb{S}\mathsf{yn}\,\mathbb{P}\mathsf{rd}\,\mathbb{W}$, the syntactic category of the regular calculus of predicates in $\mathbb{W}$. That is, understanding the regular category of left adjoints of a relational po-category is equivalent to mapping out of $\mathbb{S}\mathsf{yn}\,\mathbb{P}\mathsf{rd}\,\mathbb{W}$.

6.3 The predicates 2-functor $\mathbb{P}\mathsf{rd}\colon \mathcal{P}\mathsf{rlPoCat} \to \mathcal{R}\mathsf{gCalc}$

We are now ready to fulfil the promise we made in Examples 4.17 and 4.19: we will construct a 2-functorial assignment of regular calculi to regular categories

To begin, let us recall that Theorem 6.19 gives a 2-equivalence between the 2-category of regular categories $\mathcal{R}\mathsf{gCat}$ and that of relational po-categories $\mathcal{R}\mathsf{lPoCat}$ (Definition 6.14). With this in mind, our goal may be achieved by instead constructing a 2-functor $\mathbb{P}\mathsf{rd}\colon \mathcal{R}\mathsf{lPoCat} \to \mathcal{R}\mathsf{gCalc}$. However, it turns out that our construction of this to-be 2-functor does not depend on the presence of tabulators, and so we further factor our promised assignment as follows:

$$\mathcal{R}\mathsf{gCat} \overset{6.19}{\simeq} \mathcal{R}\mathsf{lPoCat} \rightarrowtail \mathcal{P}\mathsf{rlPoCat} \xrightarrow{\mathbb{P}\mathsf{rd}} \mathcal{R}\mathsf{gCalc}\ .$$

With that let us turn our attention to the construction of the to-be 2-functor $\mathbb{P}\mathsf{rd}$. Our first step will be to associate a right ajax po-functor to every prerelational po-category $\mathbb{R}$. In the coming proposition we will show that $\mathbb{R}(I,-)$ is right ajax, but in fact it is more generally true that $\mathbb{R}(r,-)$ is right ajax for all objects $r \in \mathrm{Ob}\,\mathbb{R}$ – though we will have no use of this fact.

Proposition 6.22. *Let* $(\mathbb{R}, \otimes, I)$ *be a prerelational po-category. The representable po-functor* $\mathbb{R}(I,-)\colon \mathbb{R} \to \mathbb{P}\mathsf{oset}$ *has a lax monoidal structure whose laxators are right adjoints. Thus* $(\mathbb{R}, \mathbb{R}(I,-))$ *supports the structure of a regular calculus.*

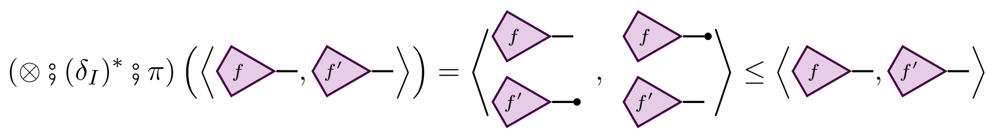
Proof. For every $r \in \mathbb{R}$, we have a poset $\mathbb{R}(I,r)$, and this is clearly functorial in r. Consider the proposed adjunctions below:

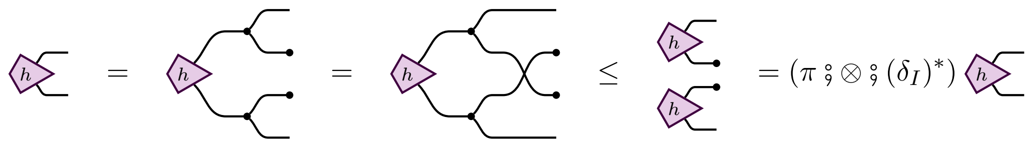
$$1 \underset{!}{\overset{\mathrm{id}_I}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbb{R}(I,I) \qquad \text{and} \qquad \mathbb{R}(I,r) \times \mathbb{R}(I,r') \underset{\pi}{\overset{\otimes\,\mathbin{;}\,(\delta_I)^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbb{R}(I, r \otimes r')\ . \tag{23}$$

The poset map labelled id_I sends 1 to the identity; that labelled ! is uniquely determined; that labelled $\otimes\,\mathbin{;}\,(\delta_I)^*$ sends $\langle f, f'\rangle$ to $f \otimes f'$ pre-composed with $\delta_I\colon I \to I \otimes I$; and that labelled π sends $h\colon I \to r \otimes r'$ to the pair $\pi(h) := \langle h \mathbin{;} (r \otimes \epsilon_{r'}) \mathbin{;} \rho_r, h \mathbin{;} (\epsilon_r \otimes r') \mathbin{;} \lambda_{r'}\rangle$. The coming contents of the proof rely on the lax comonoid homomorphism condition in Definition 6.1.

To see that the first pair of maps form an adjunction, take any $s \in \mathbb{R}(I,I)$. By definition of supply (10) we have $\epsilon_I = \mathrm{id}_I$, so lax comonoidality implies $s = s \mathbin{;} \epsilon_I \leq \epsilon_I = \mathrm{id}_I$, as desired. Let us pause here to note that the same supply conditions imply the equality of morphisms $\delta_I = {\lambda_I}^{-1} = {\rho_I}^{-1}\colon I \to I \otimes I$.

To see that the second pair of maps form an adjunction, take $\langle f, f' \rangle \in \mathbb{R}(I,r) \times \mathbb{R}(I,r')$ and $h \in \mathbb{R}(I, r \otimes r')$ and consider the below arguments, where we have used the lax comonoidality properties of (20) and the facts $\epsilon_I = \mathrm{id}_I$ and $\delta_I = \lambda_I^{-1}$ from above.





It remains to establish that $(\mathbb{R}(I,-), \mathrm{id}_I, \otimes \mathbin{⨟} (\delta_I)^*)$ assembles into a lax monoidal functor. The conditions on ! are all automatic by the terminality of $1 \in \mathbb{P}\mathsf{oset}$, and so we have reduced our claim to the assertion that the following diagram is commutative.

$$\begin{array}{ccc}
\mathbb{R}(I,(r \otimes r') \otimes r'') & \xrightarrow{\alpha^{\mathbb{R}}_*} & \mathbb{R}(I, r \otimes (r' \otimes r'')) \\
\pi\downarrow & & \downarrow\pi \\
\mathbb{R}(I, r \otimes r') \times \mathbb{R}(I, r'') & & \mathbb{R}(I,r) \times \mathbb{R}(I, r' \otimes r'') \\
\pi \times \mathrm{id}\downarrow & & \downarrow \mathrm{id} \times \pi \\
(\mathbb{R}(I,r) \times \mathbb{R}(I,r')) \times \mathbb{R}(I,r'') & \xrightarrow[\alpha^{\mathbb{P}\mathsf{oset}}]{} & \mathbb{R}(I,r) \times (\mathbb{R}(I,r') \times \mathbb{R}(I,r''))
\end{array}$$

The commutativity of this diagram on $f \in \mathbb{R}(I,(r \otimes r') \otimes r'')$ is equivalently three equalities between the corresponding components in $\mathbb{R}(I,r) \times (\mathbb{R}(I,r') \times \mathbb{R}(I,r''))$. To show all three of these equalities it is advantageous to cast $\delta_I = \lambda_I^{-1}$. In this way, the second of these equalities follows only from formal arguments on the axioms of a monoidal category, while the first and third follow from additional equations on ϵ assured by definition of supply (10). □

Thus, if $(\mathbb{R}, I, \otimes)$ is a prerelational po-category, we define

$$\mathbb{P}\mathsf{rd}(\mathbb{R}, I, \otimes) := \left(\mathbb{R} \xrightarrow{\mathbb{R}(I,-)} \mathbb{P}\mathsf{oset}\right) \text{ with } \begin{array}{c} 1 \underset{!}{\overset{\mathrm{id}}{\underset{\top}{\rightleftarrows}}} \mathbb{R}(I,I) \\[1em] \mathbb{R}(I,r) \times \mathbb{R}(I,r') \underset{\pi}{\overset{\otimes \mathbin{⨟} (\delta_I)^*}{\underset{\top}{\rightleftarrows}}} \mathbb{R}(I, r \otimes r') \end{array} \tag{24}$$

where the right ajax structure indicated on $\mathbb{R}(I,-)$ was constructed in (23) above.

Next let $(F, \varphi) \colon \mathbb{R} \to \mathbb{R}'$ be a morphism of prerelational po-categories, that is, a strong symmetric po-functor. Recall that by Lemma 6.10, F preserves the supply of $\mathbb{W}$. We define $\mathbb{P}\mathsf{rd}(F, \varphi) := (F, F^\sharp)$, where the monoidal natural transformation $F^\sharp$ below-left is given in components $F^\sharp_r\colon \mathbb{R}(I,r) \to \mathbb{R}'(I', Fr)$ as below-right.

$$\mathbb{P}\mathsf{rd}(F,\varphi) \coloneqq (F, F^\sharp) \text{ where } \qquad \begin{array}{ccc} \mathbb{R} & \xrightarrow{\mathbb{R}(I,-)} & \\ F\downarrow & \Downarrow F^\sharp & \downarrow \\ \mathbb{R}' & \xrightarrow[\mathbb{R}'(I,-)]{} & \mathbb{P}\mathsf{oset} \end{array} \tag{25}$$

$$F^\sharp_r \coloneqq \mathbb{R}(I,r) \xrightarrow{F} \mathbb{R}'(FI,Fr) \xrightarrow{(\varphi_I)^*} \mathbb{R}'(I',Fr)$$

It is straightforward to check that this definition is 1-functorial in (F,φ).

Finally, let $\alpha\colon (F,\varphi) \Rightarrow (G,\psi)$ be a 2-morphism of prerelational po-categories, that is, a left adjoint oplax-natural transformation. We wish to provide a 2-morphism of regular calculi of the form $\mathbb{P}\mathsf{rd}(\alpha) \coloneqq (\alpha, F^\sharp \leq G^\sharp)\colon\ \mathbb{P}\mathsf{rd}(F,\varphi) \Rightarrow \mathbb{P}\mathsf{rd}(G,\psi)$. To achieve this, let us note that Lemma 6.12 implies that α is, in fact, a monoidal oplax-natural transformation. As such, it remains to provide the data of a modification $F^\sharp \leq G^\sharp$ as in (17).

The required inequality $F^\sharp_r \mathbin{;} P'\alpha_r \leq G^\sharp_r$ for each $c \in \operatorname{Ob}\mathbb{R}$ follows from the monoidality of α and the oplax-naturality of α.

$$\mathbb{P}\mathsf{rd}(\alpha) \coloneqq (\alpha, F^\sharp \mathbin{;} \mathbb{R}'(I',\alpha) \leq G^\sharp) \text{ where } \qquad \begin{array}{ccc} I' & = & I' \\ \varphi_I \downarrow & \swarrow & \downarrow \psi_I \\ FI & \xrightarrow{\alpha_I} & GI \\ Ff \downarrow & \swarrow & \downarrow Gf \\ Fr & \xrightarrow[\alpha_r]{} & Gr \end{array} \tag{26}$$

Again, it is straightforward to see that this is 2-functorial by transitivity.

In summary, we have demonstrated the following.

Proposition 6.23. *The assignments of objects, morphisms and 2-morphisms given by (24), (25), and (26) assemble into a 2-functor $\mathbb{P}\mathsf{rd}\colon\ \mathcal{P}\mathsf{rlPoCat} \to \mathcal{R}\mathsf{gCalc}$.*

We will henceforth freely confuse the 2-functor $\mathbb{P}\mathsf{rd}$ with its restriction to relational po-categories

$$\mathcal{R}\mathsf{lPoCat} \rightarrowtail \mathcal{P}\mathsf{rlPoCat} \xrightarrow{\mathbb{P}\mathsf{rd}} \mathcal{R}\mathsf{gCalc}\ .$$

6.4 Graphical terms in the regular calculus of predicates

We wish to highlight, as a special case, regular calculi of the form $\mathbb{P}\mathsf{rd}\,\mathbb{R}$ where $\mathbb{R}$ is a prerelational po-category. Recall the definition of graphical terms (Definition 5.1) and observe that a graphical term in such a regular calculus comprises the data of a wiring diagram $\omega\colon r_1 \otimes \cdots \otimes r_k \to r_{\text{out}}$ in $\mathbb{R}$ as well as *morphisms* $\{\theta_i\colon I \to r_i\}_{i\in\underline{k}}$ of $\mathbb{R}$.

Note that the right ajax structure on $\mathbb{R}(I,-)$ of (23) has in particular $\texttt{true} = \mathrm{id}_I$. As such, a graphical term $(\theta_1, \ldots, \theta_k; \omega)$ with $k > 0$ represents the same predicate as the graphical term $(;\sigma \mathbin{;} (\bigotimes \theta_i) \mathbin{;} \omega)$, namely $[\![(\theta_1, \ldots, \theta_k; \omega)]\!] = \sigma \mathbin{;} (\bigotimes \theta_i) \mathbin{;} \omega$, where

$\sigma\colon I \to I^{\otimes k}$ is the appropriate symmetry. As such, we have succeeded in obtaining an equality[3] of the graphical terms $(\theta_1, \ldots, \theta_k; \omega) = (; \sigma \,⨾\, (\bigotimes \theta_i) \,⨾\, \omega)$, where the latter is merely the data of a wiring diagram $I \to r_{\mathrm{out}}$.

Of course we may read the equality $(\theta_1, \ldots, \theta_k; \omega) = (; \sigma ⨾ (\bigotimes \theta_i) ⨾ \omega)$ "the other way". Given any wiring diagram $\omega\colon I \to r_{\mathrm{out}}$ – that is, a graphical term of the form $(; \omega)$ – we may construct the graphical term $(\omega; \mathrm{id}_{r_{\mathrm{out}}})$ such that $(; \omega) = (\omega; \mathrm{id}_{r_{\mathrm{out}}})$. Graphically this is the observation that any wiring diagram with empty domain I may be equivalently re-drawn as a predicate.

Recall however that in the presence of the name-unfolding isomorphism of Lemma 3.40, the distinction between domain and codomain is fluid. With this final piece we are ready to observe that the data of graphical terms in $\mathbb{P}\mathsf{rd}\,\mathbb{R}$ is precisely the data of wiring diagrams in $\mathbb{R}$, in the following sense.

Lemma 6.24. *Let $\mathbb{R}$ be a prerelational po-category and fix $k \in \mathbb{N}$ and objects $r_1, \ldots, r_k, r_{\mathrm{out}}$ of $\mathbb{R}$. Then, with graphical terms considered in $\mathbb{P}\mathsf{rd}\,\mathbb{R}$,*

i. the set of wiring diagrams $\omega\colon \bigotimes r_i \to r_{\mathrm{out}}$ in $\mathbb{R}$ is in bijection with the set of graphical terms of the form $(; \omega'\colon I \to (\bigotimes r_i) \otimes r_{\mathrm{out}})$, mediated by the assignment $\omega \mapsto (; \omega^{\sqsubset})$,

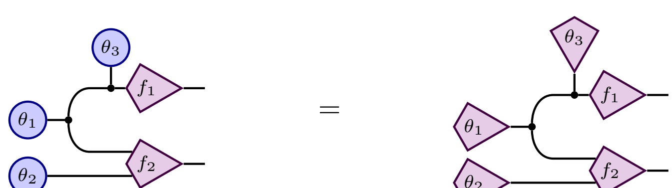
ii. the set of graphical terms $(\theta_1, \ldots, \theta_k; \omega\colon \bigotimes r_i \to r_{\mathrm{out}})$ and the set of graphical terms of the form $(; \omega'\colon I \to r_{\mathrm{out}})$ admit the following opposed functions between them which preserve the represented predicate

$$(\theta_1, \ldots, \theta_k; \omega) \mapsto (; \sigma ⨾ (\bigotimes \theta_i) ⨾ \omega)$$
$$(\omega'; \mathrm{id}_{r_{\mathrm{out}}}) \leftarrow\!\shortmid (; \omega')$$

where $\sigma\colon I \to I^{\otimes k}$ is the appropriate symmetry of $\mathbb{R}$. □

The first bijection above is a strengthening of the graphical phenomenon we identified in Example 5.10. The second correspondence, although not strictly a bijection of sets[4], lends itself readily to graphical understanding.

Example 6.25 (Graphical terms as wiring diagrams). For a given graphical term, the combination of the correspondence of Lemma 6.24 (ii) above and nesting (Proposition 5.8 (iii)) establishes that any shell containing a predicate may be re-drawn as a morphism of $\mathbb{R}$ with empty domain I without altering the represented predicate. For example, the following two graphical terms represent the same predicate.



(27)

[3] Recall that graphical terms inherit an equality relation under taking of representations, $[\![-]\!]$

[4] but rather an isomorphism of setoids

7 Comparing regular calculi and relational po-categories

In this last section we will establish several results connecting relational po-categories, and so regular categories, with regular calculi. First, in Section 7.1 below, we complement the discussion of Section 6.4 by proving that all relational po-categories are appropriately equivalent to structures comprising only the data of labelled shells connected by wires. Then, in Section 7.2, we prove the main result of this paper: the taking of left adjoints in a relational po-category may be understood by mapping out of $\mathbb{P}\mathsf{rd}\,\mathbb{W}$ in $\mathcal{R}\mathsf{gCalc}$. By appealing to the main result of the companion [cFS21], we leverage this to prove that the 2-functor LAdj is *bi-represented*. Finally, in Section 7.3, we record the main result of the companion.

7.1 Bare regular calculi from relational po-categories

Broadly construed, the goal of this paper is to establish the utility of graphical reasoning as a tool for the study of relational po-categories – and thereby for regular logic. In this section we will further cement this stance by constructing from a relational po-category $\mathbb{R}$ a *bare* regular calculus $(\bigsqcup_{\mathrm{Ob}\,\mathbb{R}} \mathbb{W}, P)$ whose syntactic po-category $\mathbb{S}\mathsf{yn}(\bigsqcup_{\mathrm{Ob}\,\mathbb{R}} \mathbb{W}, P)$ is equivalent to $\mathbb{R}$ as a relational po-category.

Although on the face of things this might seem like an unnecessary detour, because $(\bigsqcup_{\mathrm{Ob}\,\mathbb{R}} \mathbb{W}, P)$ is bare we may leverage the reduction in complexity of graphical terms of bare regular calculi (Example 5.7) as well as an explication of the particulars of this regular calculus (Example 4.20) to greatly simplify the task of working graphically in $\mathbb{R}$. That is, in this section we will establish the following outlook.

Outlook 7.1. *Every relational po-category, and so every regular category, is appropriately equivalent to a graphical calculus of labelled shells and connecting wires only, and so may be entirely understood through graphical regular logic.*

To make rigorous these claims, we will shortly prove the following.

Proposition 7.2. *Let $\mathbb{R}$ be a prerelational po-category. The supply of $\mathbb{W}$ in $\mathbb{R}$ induces a split essentially surjective strong monoidal po-functor $s^{\sqcup}\colon \bigsqcup_{\mathrm{Ob}\,\mathbb{R}} \mathbb{W} \to \mathbb{R}$. With this, the pair $\left(\bigsqcup_{\mathrm{Ob}\,\mathbb{R}} \mathbb{W},\ s^{\sqcup} \fatsemi \mathbb{R}(I,-)\right)$ supports the structure of a regular calculus. Furthermore $s^{\sqcup}$ extends to a morphism $(s^{\sqcup}, s^{\sharp})\colon (\bigsqcup_{\mathrm{Ob}\,\mathbb{R}} \mathbb{W},\ s^{\sqcup} \fatsemi \mathbb{R}(I,-)) \to \mathbb{P}\mathsf{rd}\,\mathbb{R}$ of regular calculi where $s^{\sharp}$ is invertible.*

This construction will afford us the following corollary, which when combined with our understanding of graphical terms in bare calculi in general and in $\bigsqcup_{\mathrm{Ob}\,\mathbb{R}} \mathbb{W}$ in specific gives meaning to the above outlook.

Corollary 7.3. *Let $\mathbb{R}$ be a relational po-category, then there are equivalences in $\mathcal{R}\mathsf{lPoCat}$*

$$\mathbb{R} \simeq \mathbb{S}\mathsf{yn}\,\mathbb{P}\mathsf{rd}\,\mathbb{R} \simeq \mathbb{S}\mathsf{yn}\left(\bigsqcup_{\mathrm{Ob}\,\mathbb{R}} \mathbb{W},\ s^{\sqcup} \fatsemi \mathbb{R}(I,-)\right).$$

Proof. The first equivalence follows from Theorem 7.10. To see the second, observe that the morphism $(s^{\sqcup}, s^{\sharp})\colon \big(\coprod_{\mathrm{Ob}\,\mathbb{R}} \mathbb{W},\ s^{\sqcup} \mathbin{⨾} \mathbb{R}(I,-)\big) \to \mathbb{P}\mathsf{rd}\,\mathbb{R}$ of regular calculi has $s^{\sqcup}$ split essentially surjective and $s^{\sharp}$ invertible by Proposition 7.2. Thus by Proposition 6.18, the morphism $\mathbb{S}\mathsf{yn}(s^{\sqcup}, s^{\sharp})$ of relational po-categories is an equivalence. □

Now that we have situated the importance of these combined results, let us turn our attention to the construction of the bare regular calculus.

Proof of Proposition 7.2. Let $\mathbb{R}$ be a prerelational po-category. Recall Proposition 3.29, the supply $s^{\mathbb{R}}$ of $\mathbb{W}$ in $\mathbb{R}$ induces a split essentially surjective strong monoidal po-functor $s^{\sqcup}\colon \coprod_{\mathrm{Ob}\,\mathbb{R}} \mathbb{W} \to \mathbb{R}$ which is the unique such whose pre-composition with the inclusions $\iota_r\colon \mathbb{W} \rightarrowtail \coprod_{\mathrm{Ob}\,\mathbb{R}} \mathbb{W}$ is given by $\iota_r \mathbin{⨾} s^{\sqcup} = s_r$.

With that, we will show that the pair $(\coprod_{\mathrm{Ob}\,\mathbb{R}} \mathbb{W},\ s^{\sqcup} \mathbin{⨾} \mathbb{R}(I,-))$ supports the structure of a regular calculus. First, by Lemma 3.28 and Example 6.4 we see that as $\mathbb{W}$ is prerelational, so too is $\coprod_{\mathrm{Ob}\,\mathbb{R}} \mathbb{W}$ – and so in particular this coproduct supplies $\mathbb{W}$. Next, we must show that the po-functor $s^{\sqcup} \mathbin{⨾} \mathbb{R}(I,-)\colon \coprod_{\mathrm{Ob}\,\mathbb{R}} \mathbb{W} \to \mathbb{P}\mathsf{oset}$ is right ajax. As $s^{\sqcup}$ is strong symmetric monoidal and $\mathbb{R}(I,-)$ is right ajax by Proposition 6.22, their composite is right ajax. With this we now turn our attention to extending $s^{\sqcup}$ to a morphism of regular calculi.

To give an appropriate morphism $(s^{\sqcup}, s^{\sharp})\colon (\coprod_{\mathrm{Ob}\,\mathbb{R}} \mathbb{W},\ s^{\sqcup} \mathbin{⨾} \mathbb{R}(I,-)) \to \mathbb{P}\mathsf{rd}\,\mathbb{R}$ of regular calculi we must establish that $s^{\sqcup}$ preserves the supply of $\mathbb{W}$ and provide the data of an invertible monoidal natural transformation $s^{\sharp}\colon s^{\sqcup} \mathbin{⨾} \mathbb{R}(I,-) \Rightarrow s^{\sqcup} \mathbin{⨾} \mathbb{R}(I,-)$. This latter part is easily achieved by setting $s^{\sharp} = \mathrm{id}$, and so to complete the proof it remains to establish that $s^{\sqcup}$ preserves the supply of $\mathbb{W}$. However, we know already that $\coprod_{\mathrm{Ob}\,\mathbb{R}} \mathbb{W}$ is prerelational and, as $\mathbb{R}$ is prerelational by assumption, Lemma 6.10 proves that $s^{\sqcup}$ preserves the supply of $\mathbb{W}$. □

7.2 Regular calculi recover regular categories

Using the 2-functor $\mathbb{P}\mathsf{rd}$ we may now establish a relationship tying together all of the major players. Recall that $\mathcal{R}\mathsf{gCat}$ is the 2-category of regular categories, regular functors, and natural transformations. As a first step, we may exploit the nature of regular categories and of the structure of $\mathcal{R}\mathsf{gCalc}$ to determine that the regular category $\mathsf{LAdj}\,\mathbb{R}$ of left adjoints in a relational po-category $\mathbb{R}$ may be recovered as the category of morphisms and 2-morphisms of regular calculi $\mathbb{P}\mathsf{rd}\,\mathbb{W} \to \mathbb{P}\mathsf{rd}\,\mathbb{R}$.

Theorem 7.4. *Evaluation at $\underline{1} \in \mathrm{Ob}\,\mathbb{W}$ gives rise to a pseudo-natural adjoint equivalence of 2-functors*

$$\mathcal{R}\mathsf{lPoCat} \underset{\mathsf{LAdj}}{\overset{\mathcal{R}\mathsf{gCalc}(\mathbb{P}\mathsf{rd}\,\mathbb{W},\,\mathbb{P}\mathsf{rd}\,-)}{\rightrightarrows}} \mathcal{R}\mathsf{gCat} \qquad \mathrm{ev}_{\underline{1}} \Downarrow_{\simeq} \quad .$$

As a curious consequence of Theorem 6.19, the statement that LAdj is a 2-equivalence, Theorem 7.4 above shows that the 2-category $\mathcal{R}\mathsf{gCalc}$ "knows about" all regular categories and their morphisms in a sense beyond that of our main result Theorem 7.10 – all regular categories may be found as *hom* categories in $\mathcal{R}\mathsf{gCalc}$, as opposed to somehow embedded as objects via $\mathbb{P}\mathsf{rd}$.

More still, this theorem allows us to determine a novel facet of the 2-functor LAdj: it is 2-dimensionally represented. In the sense of the theorem below then, to understand regular categories one may equivalently understand morphisms out of the syntactic po-category $\mathbb{S}\mathsf{yn}\,\mathbb{P}\mathsf{rd}\,\mathbb{W}$.

Corollary 7.5. *The* 2*-functor* $\mathsf{LAdj}\colon \mathcal{R}\mathsf{lPoCat} \to \mathcal{R}\mathsf{gCat}$ *is bi-represented by* $\mathbb{S}\mathsf{yn}\,\mathbb{P}\mathsf{rd}\,\mathbb{W}$*. That is, there is a pseudo-natural adjoint equivalence of* 2*-functors*

$$\mathcal{R}\mathsf{lPoCat}(\mathbb{S}\mathsf{yn}\,\mathbb{P}\mathsf{rd}\,\mathbb{W}, -) \simeq \mathsf{LAdj}\colon \mathcal{R}\mathsf{lPoCat} \to \mathcal{R}\mathsf{gCat} \ .$$

Proof. By composition of the pseudo-natural adjoint equivalences of Theorem 7.4 and Corollary 7.13. □

Remark 7.6. It is tempting to interpret the above result 1-categorically as some form of free-ness statement. In particular, on a regular category $\mathbb{R}$ we may apply the objects functor Ob to the equivalence $\mathcal{R}\mathsf{lPoCat}(\mathbb{S}\mathsf{yn}\,\mathbb{P}\mathsf{rd}\,\mathbb{W}, \mathbb{R}) \simeq \mathsf{LAdj}\,\mathbb{R}$. At the level of objects, the above corollary compares morphisms of relational po-categories $\mathbb{S}\mathsf{yn}\,\mathbb{P}\mathsf{rd}\,\mathbb{W} \to \mathbb{R}$ with $\mathrm{Ob}\,\mathbb{R} = \mathrm{Ob}\,\mathsf{LAdj}\,\mathbb{R}$. However, given that we began with an equivalence and not an isomorphism, taking objects merely yields functions back and forth and not a bijection. Thus, with our methods we have not in fact determined the "free relational po-category on one object".

Before we give the proof of Theorem 7.4 it will help us here and later to establish some technical lemmas. First we claim that the inclusion of {2-functors into $\mathcal{R}\mathsf{gCat}$} into {2-functors into $\mathcal{C}\mathsf{at}$} is "closed" in the following sense.

Lemma 7.7. *Let* $\mathbb{C}$ *be a* 2*-category, let* $F, G\colon \mathbb{C} \to \mathcal{C}\mathsf{at}$ *be* 2*-functors and let* $\alpha\colon F \Rightarrow G$ *be a pseudo-natural equivalence. If* G *takes values in* $\mathcal{R}\mathsf{gCat}$*, then so too does* F*.*

Proof. At the level of objects, observe that $\alpha_C\colon FC \xrightarrow{\simeq} GC$ is an equivalence of categories and so as GC is regular, so too is FC. At the level of morphisms, observe first that regular functors are stable under composition by equivalences as equivalences preserve limits and extremal epimorphisms. Moreover, in the square below where all categories are regular, and G is a regular functor, it may be verified that F is necessarily regular by virtue of being isomorphic to a regular functor.

$$\begin{array}{ccc} FC & \xrightarrow{\alpha_C} & GC \\ {\scriptstyle Ff}\downarrow & \overset{\alpha_f}{\swarrow}{\scriptstyle \cong} & \downarrow{\scriptstyle Gf} \\ FC' & \xrightarrow[\alpha_{C'}]{} & GC' \end{array}$$

As the 2-morphisms of $\mathcal{R}$gCat are simply natural transformations we have established that F takes values in $\mathcal{R}$gCat. $\square$

Secondly, as follows from the mate calculus or may be verified directly,

Lemma 7.8. *Let $F, G \colon \mathbb{C} \to \mathbb{D}$ be 2-functors and let $\alpha \colon F \Rightarrow G$ be a pseudo-natural transformation whose every object component $\alpha_C \colon FC \to GC$, for $C \in \operatorname{Ob}\mathbb{C}$, is equipped with the structure of an adjoint equivalence*

$$(\beta_C \colon GC \to FC,\ \aleph_C \colon \alpha_C \mathbin{⨾} \beta_C \Rightarrow FC,\ \beth_C \colon GC \Rightarrow \beta_C \mathbin{⨾} \alpha_C)\ .$$

Then the morphism components of α and the families $\aleph_{(-)}$ and $\beth_{(-)}$ equip β with the structure of pseudo-natural transformation, $\aleph$ and $\beth$ extend to invertible modifications $\aleph \colon \alpha \mathbin{⨾} \beta \Rrightarrow F$, $\beth \colon G \Rrightarrow \beta \mathbin{⨾} \alpha$, and $(\alpha, \beta, \aleph, \beth)$ is a pseudo-natural adjoint equivalence between G and F. $\square$

Finally we establish a universal property of the 2-functor $\mathbb{P}\mathsf{rd} \colon \mathcal{R}\mathsf{lPoCat} \to \mathcal{R}\mathsf{gCalc}$.

Lemma 7.9. *Let $(\mathbb{R}, I, \otimes)$ be a prerelational po-category. The po-functor $\mathbb{R}(I, -) \colon \mathbb{R} \to \mathbb{P}\mathsf{oset}$ is initial in the category of right ajax po-functors $\mathbb{R} \to \mathbb{P}\mathsf{oset}$ and monoidal 2-natural transformations.*

Proof. We gave $\mathbb{R}(I, -)$ a right ajax structure in Proposition 6.22, and so it remains to determine initiality. To that end, let $P \colon \mathbb{R} \to \mathbb{P}\mathsf{oset}$ be another right ajax po-functor; must to show there is a unique monoidal 2-natural transformation $s \colon \mathbb{R}(I, -) \Rightarrow P$.

The image of the laxator $\varphi_I \colon 1 \to R(I, I)$ is id_I (23), so by monoidality any such s must send id_I to the top element $s_I(\mathrm{id}) = \mathtt{true} \in P(I)$. It follows by 2-naturality and a Yoneda-style argument that the image of every element $f \in \mathbb{R}(I, r)$ is determined to be $s_r(f) \coloneqq P(f)(\mathtt{true})$:

$$\begin{array}{ccccc} & \xrightarrow{\mathrm{id}_I} & \mathbb{R}(I,I) & \xrightarrow{\mathbb{R}(I,f)} & \mathbb{R}(I,r) \\ 1 & & \downarrow{\scriptstyle s_I} & & \downarrow{\scriptstyle s_r} \\ & \xrightarrow[\mathtt{true}]{} & P(I) & \xrightarrow[P(f)]{} & P(r) \end{array} .$$

Thus any monoidal transformation must have components s_r, and the result follows when one checks that the proposed components s_r are indeed natural and monoidal. $\square$

We are now ready to give the proof of Theorem 7.4.

Proof of Theorem 7.4. By Lemma 7.7 it is enough to establish that evaluation at the object $\underline{1} \in \operatorname{Ob}\mathbb{W}$ gives a pseudo-natural adjoint equivalence between the 2-functors $\mathcal{R}\mathsf{gCalc}(\mathbb{P}\mathsf{rd}\,\mathbb{W}, \mathbb{P}\mathsf{rd}\,-)$ and LAdj taken with codomain $\mathcal{C}\mathsf{at}$. Then, by Lemma 7.8 it is enough to give a pseudo-natural transformation between the above 2-functors whose every component is an adjoint equivalence. To that end, let us begin by defining the to-be components of the pseudo-natural transformation $\mathrm{ev}_{\underline{1}} \colon \mathcal{R}\mathsf{gCalc}(\mathbb{P}\mathsf{rd}\,\mathbb{W}, \mathbb{P}\mathsf{rd}\,-) \Rightarrow \mathsf{LAdj}$.

If $\mathbb{R}$ is a relational po-category, for a morphism $(F, F^\sharp)\colon \mathsf{Prd}\,\mathbb{W} \to \mathsf{Prd}\,\mathbb{R}$ of regular calculi let us set $(\mathrm{ev}_{\underline{1}})_{\mathbb{R}}(F, F^\sharp) := F\underline{1} \in \mathrm{Ob}\,\mathbb{R} = \mathrm{Ob}\,\mathsf{LAdj}\,\mathbb{R}$. If $\alpha\colon (F, F^\sharp) \Rightarrow (G, G^\sharp)$ is a 2-morphism of $\mathcal{R}\mathsf{gCalc}$ then let $(\mathrm{ev}_{\underline{1}})_{\mathbb{R}}\alpha := \alpha_{\underline{1}} \in \mathsf{LAdj}\,\mathbb{R}(F\underline{1}, G\underline{1})$. It is straightforward to verify that this is functorial in composition of 2-morphisms in $\mathcal{R}\mathsf{gCalc}(\mathsf{Prd}\,\mathbb{W}, \mathsf{Prd}\,\mathbb{R})$, and that $(\mathrm{ev}_{\underline{1}})_{\mathbb{R}}(\mathrm{id}_{(F,F^\sharp)})$ is the identity so that $(\mathrm{ev}_{\underline{1}})_{\mathbb{R}}$ is a functor. Finally, through a simple expansion of definitions, the components of $\mathrm{ev}_{\underline{1}}$ just constructed may be seen to satisfy the two conditions of 2-naturality. Thus $\mathrm{ev}_{\underline{1}}$ is a 2-natural transformation.

It remains then, for a fixed but arbitrary relational po-category $\mathbb{R}$, to supply the structure of an adjoint equivalence on $(\mathrm{ev}_{\underline{1}})_{\mathbb{R}}$. To do so we will begin with the opposing functor $(-)^{\otimes}_{\mathbb{R}}\colon \mathsf{LAdj}\,\mathbb{R} \to \mathcal{R}\mathsf{gCalc}(\mathsf{Prd}\,\mathbb{W}, \mathsf{Prd}\,\mathbb{R})$.

On objects $r \in \mathrm{Ob}\,\mathbb{R} = \mathrm{Ob}\,\mathsf{LAdj}\,\mathbb{R}$, recall that the strong monoidal po-functor $s_r(-)\colon \mathbb{W} \to \mathbb{R}$ determined by the supply of $\mathbb{W}$ in $\mathbb{R}$ is supply preserving by Lemma 3.36. Thus let us set $r^{\otimes} := (s_r(-), \exists!)$ as the pair of $s_r(-)\colon \mathbb{W} \to \mathbb{R}$ and the unique monoidal natural transformation $\mathbb{W}(\underline{0}, -) \Rightarrow \mathbb{R}(I, s_r(-))$ guaranteed by Lemma 7.9. On morphisms $f\colon r \to t$ of $\mathsf{LAdj}\,\mathbb{R}$, consider the family $(f^{\otimes})_{\underline{n}} := f^{\otimes n}\colon s_r(\underline{n}) \to s_t(\underline{n})$ of left adjoints in $\mathbb{R}$. We contend that these data together form a monoidal left adjoint oplax-natural transformation $f^{\otimes}\colon s_r \Rightarrow s_t$. Indeed monoidality is evident by definition, and because $\mathbb{R}$ is relational the components $f^{\otimes n}$ commute strictly with all left adjoints of $\mathbb{W}$ ([FS19b, Proposition 6.9 (iv)]). Thus, by a mate argument the components $f^{\otimes n}$ are oplax-natural on right adjoints of $\mathbb{W}$, and thus oplax-natural on all morphisms of $\mathbb{W}$ for $\mathbb{W}$ is generated by left and right adjoints (Section 3.1). However, to cast $f^{\otimes}$ as a 2-morphism $r^{\otimes} \Rightarrow t^{\otimes}$ of $\mathcal{R}\mathsf{gCalc}$ we must also verify that the modification condition (17) holds. Closer examination of this requirement, however, reveals it to be a particular case of the already established oplax-naturality of $f^{\otimes}$. Lastly, it is straightforward to see that $(-)^{\otimes}$ is functorial.

Next let us turn our attention to the unit and co-unit of the to-be adjoint equivalence mediated by $(\mathrm{ev}_{\underline{1}})_{\mathbb{R}}$ and $(-)^{\otimes}_{\mathbb{R}}$. Observe that $(-)^{\otimes}_{\mathbb{R}} \mathbin{;} (\mathrm{ev}_{\underline{1}})_{\mathbb{R}}$ is the identity on $\mathsf{LAdj}\,\mathbb{R}$, so we may take the unit as $\aleph := \mathrm{id}_{\mathsf{LAdj}\,\mathbb{R}}$. Then, if $(F, F^\sharp)\colon \mathsf{Prd}\,\mathbb{W} \to \mathsf{Prd}\,\mathbb{R}$ is a morphism in $\mathcal{R}\mathsf{gCalc}$, the component $\beth_{(F,F^\sharp)}$ of the co-unit $\beth\colon (\mathrm{ev}_{\underline{1}})_{\mathbb{R}} \mathbin{;} (-)^{\otimes}_{\mathbb{R}} \Rightarrow \mathrm{id}_{\mathcal{R}\mathsf{gCalc}(\dots)}$ must be an invertible 2-morphism $(F\underline{1})^{\otimes} \Rightarrow (F, F^\sharp)$ in $\mathcal{R}\mathsf{gCalc}$. For the underlying invertible monoidal oplax-natural transformation $s_{F\underline{1}}(-) \Rightarrow F$ we may choose the canonical isomorphisms $F(\underline{1})^{\otimes n} \to F(\underline{n})$ comprising strongators of F, and then the modification condition (17) becomes a comparison between two monoidal natural transformations $W(\underline{0}, -) \Rightarrow F$. But, by Lemma 7.9 these must coincide and the condition holds. Finally, the naturality of the components of $\beth$ amounts to the monoidality property of the underlying monoidal left adjoint oplax-natural transformations of the 2-morphisms of $\mathcal{R}\mathsf{gCalc}$.

To complete the proof it remains only to verify that the triangle equalities hold for $\aleph$ and $\beth$, a task simplified by the fact that $\aleph = \mathrm{id}_{\mathsf{LAdj}\,\mathbb{R}}$. These identities thus amount to the observations that $\beth_{r^{\otimes}} = \mathrm{id}_{r^{\otimes}}$ and $(\mathrm{ev}_{\underline{1}})_{\mathbb{R}}(\beth_{(F,F^\sharp)}) = \mathrm{id}_{(\mathrm{ev}_{\underline{1}})_{\mathbb{R}}(F,F^\sharp)}$. □

7.3 Main results of the companion

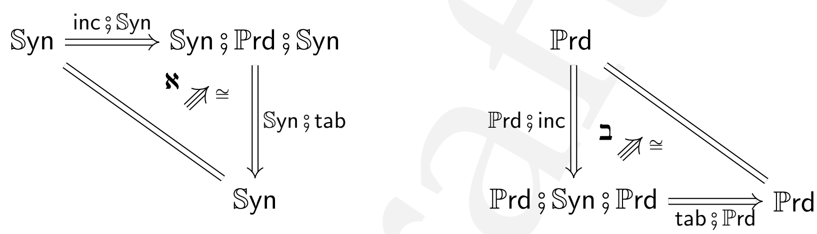
To end this paper and introduce the companion, we record here the main result of the work thus far on graphical regular logic. We have already described opposed 2-functors

$$\mathbb{S}\mathsf{yn}\colon\ \mathcal{R}\mathsf{gCalc} \leftrightarrows \mathcal{R}\mathsf{lPoCat}\ :\mathbb{P}\mathsf{rd}$$

in Sections 5.3 and 6.3. In the companion paper we furnish additionally pseudo-natural transformations

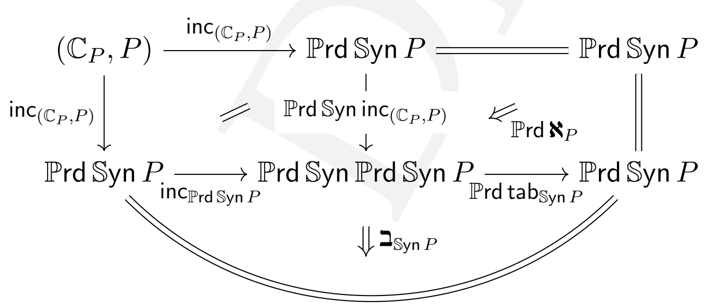
, (28)

where tab carries the structure of a pseudo-natural adjoint equivalence and is given by taking tabulators in a certain capacity. We additionally construct invertible modifications

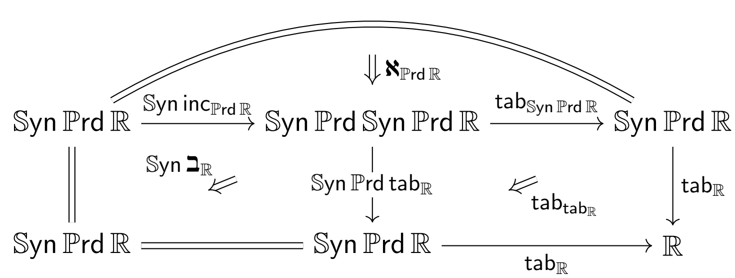


, (29)

which together satisfy the so-called "swallow-tail identities":



$= \mathrm{id}_{\mathsf{inc}_{(\mathbb{C}_P,P)}}$ (30)

$\mathrm{id}_{\mathsf{tab}_{\mathbb{R}}}$ =


(31)

Our central thesis is that, taken together, these data and properties afford a rich comparison of the 2-**category theory** of relational po-categories with that of regular calculi. More precisely,

Theorem 7.10 ([cFS21, ???]). *The data of* (28) *and* (29)*, and the properties* (30) *and* (31) *provide the* 2*-functors* $\mathbb{S}\mathsf{yn}\colon \mathcal{R}\mathsf{gCalc} \to \mathcal{R}\mathsf{lPoCat}$ *and* $\mathbb{P}\mathsf{rd}\colon \mathcal{R}\mathsf{lPoCat} \to \mathcal{R}\mathsf{gCalc}$ *with the structure of a a bi-adjunction* $\mathbb{S}\mathsf{yn} \dashv_{\mathrm{bi}} \mathbb{P}\mathsf{rd}$*. Moreover, as the co-unit* tab *is part of an adjoint equivalence this bi-adjunction is pseudo-reflection of* $\mathcal{R}\mathsf{lPoCat}$ *into* $\mathcal{R}\mathsf{gCalc}$*.*

For a detailed account of this theorem and the relevant omitted constructions we direct the reader to the companion paper [cFS21].

By the equivalence of relational po-categories and regular categories, any comparison of the 2-category theory of regular calculi with that of relational categories extends to a comparison with the 2-category theory of regular categories.

Corollary 7.11. *The* 2*-category* $\mathcal{R}\mathsf{gCat}$ *of regular categories is pseudo-reflective in* $\mathcal{R}\mathsf{gCalc}$*.*

Proof. Although we will not produce a proof here, it may be verified that a 2-equivalence composed with a pseudo-reflection is again a pseudo-reflection. Thus the result follows from Theorems 6.19 and 7.10. □

We record here finally the following folk-lore lemma which affords us a convenient recasting of Theorem 7.10.

Lemma 7.12. *Given a pair of opposed pseudo-functors* $L\colon \mathbb{C} \leftrightarrows \mathbb{D} : R$*, the structure of a bi-adjunction* $L \dashv_{\mathrm{bi}} R$ *on* L *and* R *is equivalently the structure of a pseudo-natural adjoint equivalence of the pseudo-functors* $\mathbb{D}(L-,-) \simeq \mathbb{C}(-,R-)\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{D} \to \mathcal{C}\mathsf{at}$*. Additionally, the co-unit is an adjoint equivalence if and only if the image of* $\mathrm{id}_{RC} \in \mathbb{C}(RD, RD)$ *is an equivalence* $LRD \simeq D$ *in* $\mathbb{D}$ *for each* $D \in \mathrm{Ob}\,\mathbb{D}$*.* □

Corollary 7.13. *There is a pseudo-natural adjoint equivalence of* 2*-functors*

$$\mathcal{R}\mathsf{lPoCat}(\mathbb{S}\mathsf{yn}\,-,-) \simeq \mathcal{R}\mathsf{gCalc}(-,\mathbb{P}\mathsf{rd}\,-)\colon\ \mathcal{R}\mathsf{gCalc}^{\mathrm{op}} \times \mathcal{R}\mathsf{lPoCat} \to \mathcal{C}\mathsf{at}$$

such that the image of $\mathrm{id}_{\mathbb{P}\mathsf{rd}\,\mathbb{R}} \in \mathcal{R}\mathsf{gCalc}(\mathbb{P}\mathsf{rd}\,\mathbb{R}, \mathbb{P}\mathsf{rd}\,\mathbb{R})$ *is an equivalence* $\mathbb{S}\mathsf{yn}\,\mathbb{P}\mathsf{rd}\,\mathbb{R} \simeq \mathbb{R}$ *for each relational po-category* $\mathbb{R}$*.* □

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