Title:	*Temporal Logic and Model Checking*
Lecturer:	**Mike Gordon**
Class:	**Computer Science Tripos, Part II**
Term:	**Lent Term 2015**
Lecture 1:	**10:00 on Thu, 15 Jan, 2015**
Lecture 2:	**10:00 on Tue, 20 Jan, 2015**
Lecture 3:	**10:00 on Thu, 22 Jan, 2015**
Lecture 4:	**10:00 on Tue, 27 Jan, 2015**
Lecture 5:	**10:00 on Thu, 29 Jan, 2015**
Lecture 6:	**10:00 on Tue, 03 Feb, 2015**
Lecture 7:	**10:00 on Thu, 05 Feb, 2015**
Lecture 8:	**10:00 on Tue, 10 Feb, 2015**
Location:	**Lecture Theatre 2, WGB**
Duration:	**Eight lectures**

Document created December 1, 2014

Topics and corresponding slides

Temporal Logic and Model Checking

- Model
  - mathematical structure extracted from hardware or software
- Temporal logic
  - provides a language for specifying functional properties
- Model checking
  - checks whether a given property holds of a model

- Model checking is a kind of static verification
  - dynamic verification is simulation (HW) or testing (SW)

Models

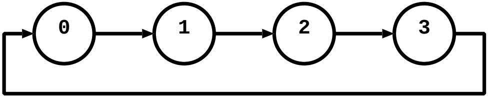
- A model is (for now) specified by a pair (S, R)
  - S is a set of *states*
  - R is a *transition relation*

- Models will get more components later
  - (S, R) also called a transition system

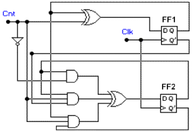
- $R\ s\ s'$ means s' can be reached from s in one step
  - here $R : S \to (S \to \mathbb{B})$ (where $\mathbb{B} = \{true, false\}$)
  - more conventional to have $R \subseteq S \times S$, which is equivalent
  - i.e. $R_{\text{(this course)}}\ s\ s' \Leftrightarrow (s, s') \in R_{\text{(some textbooks)}}$

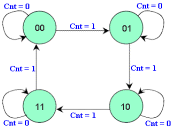
A simple example model

- A simple model: $(\underbrace{\{0,1,2,3\}}_{S}, \underbrace{\lambda n\ n'.\ n' = n{+}1(mod\ 4)}_{R})$
  - where "$\lambda x.\ \cdots x \cdots$" is the function mapping x to $\cdots x \cdots$
  - so $R\ n\ n' = (n' = n{+}1(mod\ 4))$
  - e.g. $R\ 0\ 1 \wedge R\ 1\ 2 \wedge R\ 2\ 3 \wedge R\ 3\ 0$



- Might be extracted from:





[Acknowledgement: http://eelab.usyd.edu.au/digital_tutorial/part3/t-diag.htm]

DIV: a software example

- Perhaps a familiar program:

```
0:   R:=X;
1:   Q:=0;
2:   WHILE Y≤R DO
3:   (R:=R-Y;
4:    Q:=Q+1)
5:
```

- State (pc, x, y, r, q)
  - $pc \in \{0, 1, 2, 3, 4, 5\}$ program counter
  - $x, y, r, q \in \mathbb{Z}$ are the values of X, Y, R, Q
- Model $(S_{\texttt{DIV}}, R_{\texttt{DIV}})$ where:

$$S_{\texttt{DIV}} = [0..5] \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \qquad \text{(where } [m..n] = \{m, m{+}1, \ldots, n\}\text{)}$$

$$\begin{array}{l}
R_{\texttt{DIV}}\ (pc, x, y, r, q)\ (pc', x', y', r', q') = \\
\quad (pc = 0) \Rightarrow ((pc', x', y', r', q') = (1, x, y, x, q)) \qquad \wedge \\
\quad (pc = 1) \Rightarrow ((pc', x', y', r', q') = (2, x, y, r, 0)) \qquad \wedge \\
\quad (pc = 2) \Rightarrow ((pc', x', y', r', q') = \\
\qquad\qquad\qquad \mathit{if}\ y{\leq}r\ \mathit{then}\ (3, x, y, r, q)\ \mathit{else}\ (5, x, y, r, q)) \qquad \wedge \\
\quad (pc = 3) \Rightarrow ((pc', x', y', r', q') = (4, x, y, (r{-}y), q)) \qquad \wedge \\
\quad (pc = 4) \Rightarrow ((pc', x', y', r', q') = (2, x, y, r, (q{+}1)))
\end{array}$$

Deriving a transition relation from a state machine

- State machine transition function: $\delta \ : \ \mathit{Inp} \times \mathit{Mem} \to \mathit{Mem}$
  - Inp is a set of inputs
  - Mem is a memory (set of storable values)

- Model: (S_δ, R_δ) where:

  $$S_\delta = \mathit{Inp} \times \mathit{Mem}$$
  $$R_\delta\ (i, m)\ (i', m') \ = \ (m' \ = \ \delta(i, m))$$

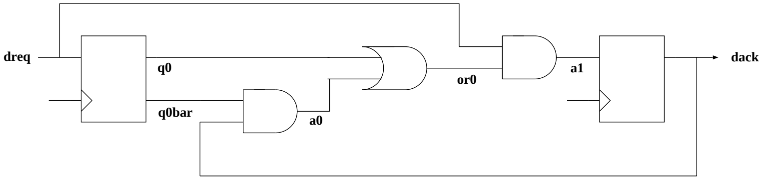
  and

  - i' arbitrary: determined by environment not by machine
  - m' determined by input and current state of machine

- Deterministic machine, non-deterministic transition relation
  - inputs unspecified (determined by environment)
  - so called "input non-determinism"

RCV: a state machine specification of a circuit

- Part of a handshake circuit:



- Input: $dreq$, Memory: $(q0, dack)$
- Relationships between Boolean values on wires:

$$
\begin{array}{lcl}
q0bar & = & \neg q0 \\
a0 & = & q0bar \wedge dack \\
or0 & = & q0 \vee a0 \\
a1 & = & dreq \wedge or0
\end{array}
$$

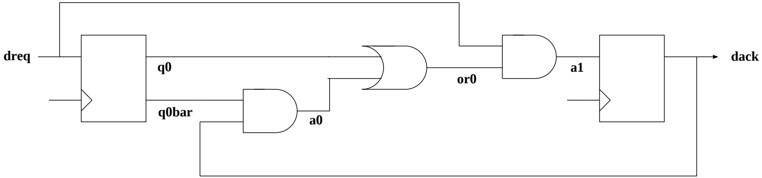
- State machine: $\delta_{\mathrm{RCV}} : \mathbb{B} \times (\mathbb{B}\times\mathbb{B}) \to (\mathbb{B}\times\mathbb{B})$

$$\delta_{\mathrm{RCV}}\ (\underbrace{dreq,}_{Inp}\ \underbrace{(q0, dack)}_{Mem}) = (dreq,\ dreq \wedge (q0 \vee (\neg q0 \wedge dack)))$$

- RTL model – could have lower level model with clock edges

RCV: a model of the circuit

- Circuit from previous slide:



- State represented by a triple of Booleans $(dreq, q0, dack)$
- By De Morgan Law: $q0 \lor (\neg q0 \land dack) = q0 \lor dack$
- Hence δ_{RCV} corresponds to model $(S_{\mathrm{RCV}}, R_{\mathrm{RCV}})$ where:

$$S_{\mathrm{RCV}} = \mathbb{B} \times \mathbb{B} \times \mathbb{B}$$

$$\begin{aligned} R_{\mathrm{RCV}}\ & (dreq, q0, dack)\ (dreq', q0', dack') = \\ & (q0' = dreq) \land (dack' = (dreq \land (q0 \lor dack))) \end{aligned}$$

[Note: we are identifying $\mathbb{B} \times \mathbb{B} \times \mathbb{B}$ with $\mathbb{B} \times (\mathbb{B} \times \mathbb{B})$]

Some comments

- $R_{\texttt{RCV}}$ is non-deterministic and total
  - $R_{\texttt{RCV}}\ (1,1,1)\ (0,1,1)$ and $R_{\texttt{RCV}}\ (1,1,1)\ (1,1,1)$
    (where $1 = \mathit{true}$ and $0 = \mathit{false}$)
  - $R_{\texttt{RCV}}\ (\mathit{dreq}, q0, \mathit{dack})\ (\mathit{dreq}', \mathit{dreq}, (\mathit{dreq} \wedge (q0 \vee \mathit{dack})))$
- $R_{\texttt{DIV}}$ is deterministic and partial
  - at most one successor state
  - no successor when $pc = 5$
- Non-deterministic models are very common, e.g. from:
  - asynchronous hardware
  - parallel software (more than one thread)
- Can extend any transition relation R to be total:

$$\begin{aligned} R_{total}\ s\ s' &= \mathit{if}\ (\exists s''.\ R\ s\ s'')\ \mathit{then}\ R\ s\ s'\ \mathit{else}\ (s' = s) \\ &= R\ s\ s' \vee (\neg(\exists s''.\ R\ s\ s'') \wedge (s' = s)) \end{aligned}$$

  - sometimes totality required
    (e.g. in the book *Model Checking* by Clarke et. al)

JM1: a non-deterministic software example

- From Jhala and Majumdar's tutorial:

Thread 1	Thread 2
`0: IF LOCK=0 THEN LOCK:=1;`	`0: IF LOCK=0 THEN LOCK:=1;`
`1: X:=1;`	`1: X:=2;`
`2: IF LOCK=1 THEN LOCK:=0;`	`2: IF LOCK=1 THEN LOCK:=0;`
`3:`	`3:`

- Two program counters, state: $(pc_1, pc_2, lock, x)$

$$
\begin{array}{l}
S_{\mathtt{JM1}} \ = \ [0..3] \times [0..3] \times \mathbb{Z} \times \mathbb{Z} \\
R_{\mathtt{JM1}}\ (0, pc_2, 0, x) \quad (1, pc_2, 1, x) \\
R_{\mathtt{JM1}}\ (1, pc_2, lock, x) \ (2, pc_2, lock, 1) \\
R_{\mathtt{JM1}}\ (2, pc_2, 1, x) \quad (3, pc_2, 0, x) \\
R_{\mathtt{JM1}}\ (pc_1, 0, 0, x) \quad (pc_1, 1, 1, x) \\
R_{\mathtt{JM1}}\ (pc_1, 1, lock, x) \ (pc_1, 2, lock, 2) \\
R_{\mathtt{JM1}}\ (pc_1, 2, 1, x) \quad (pc_1, 3, 0, x)
\end{array}
$$

- Not-deterministic:

$$
\begin{array}{l}
R_{\mathtt{JM1}}\ (0, 0, 0, x) \ (1, 0, 1, x) \\
R_{\mathtt{JM1}}\ (0, 0, 0, x) \ (0, 1, 1, x)
\end{array}
$$

- Not so obvious that $R_{\mathtt{JM1}}$ is a correct model

Atomic properties (properties of states)

- Atomic properties are true or false of individual states
  - an atomic property p is a function $p : S \to \mathbb{B}$
  - can also be regarded as a subset of state: $p \subseteq S$
- Example atomic properties of `RCV` (where $1 = \mathit{true}$ and $0 = \mathit{false}$)

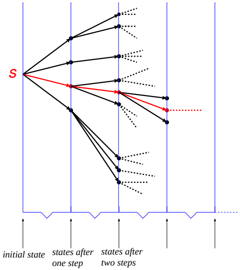
$$\begin{array}{lcl}
\texttt{Dreq}(\mathit{dreq}, q0, \mathit{dack}) & = & (\mathit{dreq} = 1) \\
\texttt{NotQ0}(\mathit{dreq}, q0, \mathit{dack}) & = & (q0 = 0) \\
\texttt{Dack}(\mathit{dreq}, q0, \mathit{dack}) & = & (\mathit{dack} = 1) \\
\texttt{NotDreqAndQ0}(\mathit{dreq}, q0, \mathit{dack}) & = & (\mathit{dreq}{=}0) \wedge (q0{=}1)
\end{array}$$

- Example atomic properties of `DIV`

$$\begin{array}{lcl}
\texttt{AtStart}\ (pc, x, y, r, q) & = & (pc = 0) \\
\texttt{AtEnd}\ (pc, x, y, r, q) & = & (pc = 5) \\
\texttt{InLoop}\ (pc, x, y, r, q) & = & (pc \in \{3, 4\}) \\
\texttt{YleqR}\ (pc, x, y, r, q) & = & (y \leq r) \\
\texttt{Invariant}\ (pc, x, y, r, q) & = & (x = r + (y \times q))
\end{array}$$

Model behaviour viewed as a computation tree

- Atomic properties are true or false of individual states
- General properties are true or false of whole behaviour
- Behaviour of (S, R) starting from $s \in S$ as a tree:



- A path is shown in red
- Properties may look at all paths, or just a single path
  - CTL: Computation Tree Logic (all paths from a state)
  - LTL: Linear Temporal Logic (a single path)

Paths

- A path of (S, R) is represented by a function $\pi : \mathbb{N} \to S$
  - $\pi(i)$ is the ith element of π (first element is $\pi(0)$)
  - might sometimes write $\pi\ i$ instead of $\pi(i)$
  - $\pi{\downarrow}i$ is the i-th tail of π so $\pi{\downarrow}i(n) = \pi(i+n)$
  - successive states in a path must be related by R

- Path $R\ s\ \pi$ is true if and only if π is a path starting at s:

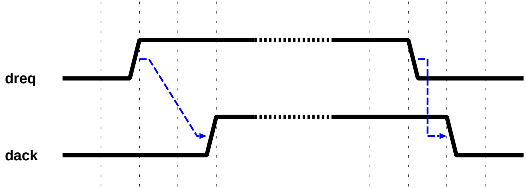
$$\mathrm{Path}\ R\ s\ \pi \ = \ (\pi(0) = s)\ \wedge\ \forall i.\ R\ (\pi(i))\ (\pi(i{+}1))$$

where:

$$\mathrm{Path} : \underbrace{(S \to S \to \mathbb{B})}_{\text{transition relation}} \to \underbrace{S}_{\text{initial state}} \to \underbrace{(\mathbb{N} \to S)}_{\text{path}} \to \mathbb{B}$$

RCV: example hardware properties

- Consider this timing diagram:



- Two handshake properties representing the diagram:
  - following a rising edge on `dreq`, the value of `dreq` remains 1 (i.e. *true*) until it is acknowledged by a rising edge on `dack`
  - following a falling edge on `dreq`, the value on `dreq` remains 0 (i.e. *false*) until the value of `dack` is 0
- A property language is used to formalise such properties

DIV: example program properties

```
0:   R:=X;
1:   Q:=0;
2:   WHILE Y≤R DO
3:   (R:=R-Y;
4:    Q:=Q+1)
5:
```

$$
\begin{array}{lcl}
\texttt{AtStart}\,(pc, x, y, r, q) & = & (pc = 0) \\
\texttt{AtEnd}\,(pc, x, y, r, q) & = & (pc = 5) \\
\texttt{InLoop}\,(pc, x, y, r, q) & = & (pc \in \{3, 4\}) \\
\texttt{YleqR}\,(pc, x, y, r, q) & = & (y \leq r) \\
\texttt{Invariant}\,(pc, x, y, r, q) & = & (x = r + (y \times q))
\end{array}
$$

- Example properties of the program DIV.
  - on every execution if AtEnd is true then Invariant is true and YleqR is not true
  - on every execution there is a state where AtEnd is true
  - on any execution if there exists a state where YleqR is true then there is also a state where InLoop is true
- Compare these with what is expressible in Hoare logic
  - execution: a path starting from a state satisfying AtStart

Recall `JM1`: a non-deterministic program example

Thread 1	Thread 2
`0:   IF LOCK=0 THEN LOCK:=1;`	`0:   IF LOCK=0 THEN LOCK:=1;`
`1:   X:=1;`	`1:   X:=2;`
`2:   IF LOCK=1 THEN LOCK:=0;`	`2:   IF LOCK=1 THEN LOCK:=0;`
`3:`	`3:`

$$
\begin{array}{lll}
R_{\mathtt{JM1}} & (0, pc_2, 0, x) & (1, pc_2, 1, x) \\
R_{\mathtt{JM1}} & (1, pc_2, lock, x) & (2, pc_2, lock, 1) \\
R_{\mathtt{JM1}} & (2, pc_2, 1, x) & (3, pc_2, 0, x) \\
R_{\mathtt{JM1}} & (pc_1, 0, 0, x) & (pc_1, 1, 1, x) \\
R_{\mathtt{JM1}} & (pc_1, 1, lock, x) & (pc_1, 2, lock, 2) \\
R_{\mathtt{JM1}} & (pc_1, 2, 1, x) & (pc_1, 3, 0, x)
\end{array}
$$

- An atomic property:
  - $\mathtt{NotAt11}(pc_1, pc_2, lock, x) = \neg((pc_1 = 1) \land (pc_2 = 1))$
- A non-atomic property:
  - all states reachable from $(0, 0, 0, 0)$ satisfy `NotAt11`
  - this is an example of a reachability property

State satisfying `NotAt11` unreachable from $(0, 0, 0, 0)$

```
Thread 1                              Thread 2
0:  IF LOCK=0 THEN LOCK:=1;           0:  IF LOCK=0 THEN LOCK:=1;
1:  X:=1;                             1:  X:=2;
2:  IF LOCK=1 THEN LOCK:=0;           2:  IF LOCK=1 THEN LOCK:=0;
3:                                    3:
```

$R_{\mathrm{JM1}}\ (0, pc_2, 0, x)$	$(1, pc_2, 1, x)$	$R_{\mathrm{JM1}}\ (pc_1, 0, 0, x)$	$(pc_1, 1, 1, x)$
$R_{\mathrm{JM1}}\ (1, pc_2, lock, x)$	$(2, pc_2, lock, 1)$	$R_{\mathrm{JM1}}\ (pc_1, 1, lock, x)$	$(pc_1, 2, lock, 2)$
$R_{\mathrm{JM1}}\ (2, pc_2, 1, x)$	$(3, pc_2, 0, x)$	$R_{\mathrm{JM1}}\ (pc_1, 2, 1, x)$	$(pc_1, 3, 0, x)$

- `NotAt11` $(pc_1, pc_2, lock, x) = \neg((pc_1 = 1) \wedge (pc_2 = 1))$
- Can only reach $pc_1 = 1 \ \wedge\ pc_2 = 1$ via:

  $R_{\mathrm{JM1}}\ (0, pc_2, 0, x)\quad (1, pc_2, 1, x)$ i.e. a step $R_{\mathrm{JM1}}\ (0, 1, 0, x)\quad (1, 1, 1, x)$

  $R_{\mathrm{JM1}}\ (pc_1, 0, 0, x)\quad (pc_1, 1, 1, x)$ i.e. a step $R_{\mathrm{JM1}}\ (1, 0, 0, x)\quad (1, 1, 1, x)$
- But:

  $R_{\mathrm{JM1}}\ (pc_1, pc_2, lock, x)\ (pc_1', pc_2', lock', x')\ \wedge\ pc_1'{=}0\ \wedge\ pc_2'{=}1\ \Rightarrow\ lock'{=}1$

  $\wedge$

  $R_{\mathrm{JM1}}\ (pc_1, pc_2, lock, x)\ (pc_1', pc_2', lock', x')\ \wedge\ pc_1'{=}1\ \wedge\ pc_2'{=}0\ \Rightarrow\ lock'{=}1$
- So can never reach $(0, 1, 0, x)$ or $(1, 0, 0, x)$
- So can't reach $(1, 1, 1, x)$, hence never $(pc_1 = 1) \wedge (pc_2 = 1)$
- Hence all states reachable from $(0, 0, 0, 0)$ satisfy `NotAt11`

Reachability

- $R\ s\ s'$ means s' reachable from s in one step
- $R^n\ s\ s'$ means s' reachable from s in n steps

$$\begin{array}{lcl} R^0\ s\ s' & = & (s = s') \\ R^{n+1}\ s\ s' & = & \exists s''.\ R\ s\ s'' \wedge R^n\ s''\ s' \end{array}$$

- $R^*\ s\ s'$ means s' reachable from s in finite steps

$$R^*\ s\ s' = \exists n.\ R^n\ s\ s'$$

- Note: $R^*\ s\ s' \Leftrightarrow \exists \pi\ n.\ \mathrm{Path}\ R\ s\ \pi \wedge (s' = \pi(n))$
- The set of states reachable from s is $\{s' \mid R^*\ s\ s'\}$
- Verification problem: all states reachable from s satisfy p
  - verify truth of $\forall s'.\ R^*\ s\ s' \Rightarrow p(s')$
  - e.g. all states reachable from $(0,0,0,0)$ satisfy `NotAt11`
  - i.e. $\forall s'.\ R^*_{\mathtt{JM1}}\ (0,0,0,0)\ s' \Rightarrow \mathtt{NotAt11}(s')$

Models and model checking

- Assume a model (S, R)
- Assume also a set $S_0 \subseteq S$ of initial states
- Assume also a set AP of atomic properties
- Assume a labeling function $L : S \to \mathcal{P}(AP)$
  - $p \in L(s)$ means "s labelled with p" or "p true of s"
  - previously properties were functions $p : S \to \mathbb{B}$
  - now $p \in AP$ is distinguished from $\lambda s.\ p \in L(s)$
  - assume $\mathtt{T}, \mathtt{F} \in AP$ with forall s: $\mathtt{T} \in L(s)$ and $\mathtt{F} \notin L(s)$
- A *Kripke structure* is a tuple (S, S_0, R, L)
  - often the term "model" is used for a Kripke structure
  - i.e. a model is (S, S_0, R, L) rather than just (S, R)
- Model checking computes whether $(S, S_0, R, L) \models \phi$
  - ϕ is a property expressed in a property language
  - informally $M \models \phi$ means "wff ϕ is true in model M"

Minimal property language: ϕ is $\mathbf{AG}\,p$ where $p \in AP$

- Consider properties ϕ of form $\mathbf{AG}\,p$ where $p \in AP$
  - "**AG**" stands for "Always Globally"
- Assume $M = (S, S_0, R, L)$
- Reachable states of M are $\{s' \mid \exists s \in S_0.\ R^*\ s\ s'\}$
  - i.e. the set of states reachable from an initial state
  - define $\text{Reachable}\ M = \{s' \mid \exists s \in S_0.\ R^*\ s\ s'\}$
- $M \models \mathbf{AG}\,p$ means p true of all reachable states of M
- If $M = (S, S_0, R, L)$ then $M \models \phi$ formally defined by:

$$\boxed{M \models \mathbf{AG}\,p \Leftrightarrow \forall s'.\ s' \in \text{Reachable}\ M \Rightarrow p \in L(s')}$$

Model checking $M \models \mathbf{AG}\, p$

- $M \models \mathbf{AG}\, p \Leftrightarrow \forall s'.\ s' \in \text{Reachable } M \Rightarrow p \in L(s')$
  $\Leftrightarrow \text{Reachable } M \subseteq \{s' \mid p \in L(s')\}$

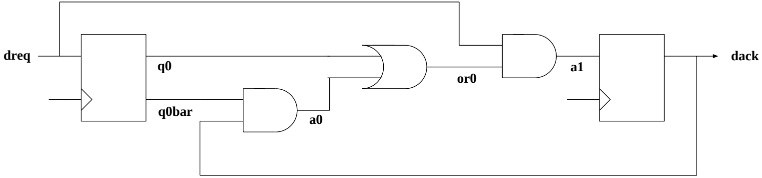
  so:
  - compute Reachable M i.e. compute $\{s' \mid \exists s \in S_0.\ R^*\ s\ s'\}$
  - check p true of all its members
- Let $\mathcal{S} = \{s' \mid \exists s \in S_0.\ R^*\ s\ s'\}$
- Compute $\mathcal{S}$ iteratively: $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_n \cup \cdots$
  - i.e. $\mathcal{S} = \bigcup_{n=0}^{\infty} \mathcal{S}_n$
  - where: $\mathcal{S}_0 = S_0$ (set of initial states)
  - and inductively: $\mathcal{S}_{n+1} = \mathcal{S}_n \cup \{s' \mid \exists s \in \mathcal{S}_n \wedge R\ s\ s'\}$
- Clearly $\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{S}_n \subseteq \cdots$
- Hence if $\mathcal{S}_{\mathrm{m}} = \mathcal{S}_{\mathrm{m}+1}$ then $\mathcal{S} = \mathcal{S}_{\mathrm{m}}$
- Algorithm: compute $\mathcal{S}_0, \mathcal{S}_1, \ldots,$ until no change;
  check all members of computed set labelled with p

compute $\mathcal{S}_0$, $\mathcal{S}_1$, ... , until no change;
check p holds of all members of computed set

- Does the algorithm terminate?
  - yes, if set of states is finite, because then no infinite chains:
    $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_n \subset \cdots$
- How to represent $\mathcal{S}_0$, $\mathcal{S}_1$, ... ?
  - explicitly (e.g. lists or something more clever)
  - symbolic expression
- Huge literature on calculating set of reachable states

Example: RCV

- Recall the handshake circuit:



- State represented by a triple of Booleans $(dreq, q0, dack)$
- A model of RCV is M_{RCV} where:

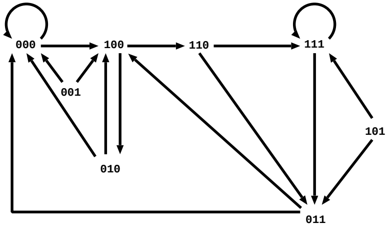
  $M = (S_{\mathrm{RCV}}, \{(1,1,1)\}, R_{\mathrm{RCV}}, L_{\mathrm{RCV}})$

  and

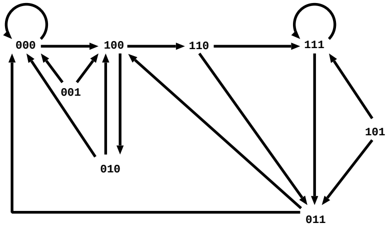
  $$\begin{array}{l} R_{\mathrm{RCV}}\ (dreq, q0, dack)\ (dreq', q0', dack') = \\ \quad (q0' = dreq) \wedge (dack' = (dreq \wedge (q0 \vee dack))) \end{array}$$

$\texttt{RCV}$ state transition diagram

- Possible states for $\texttt{RCV}$:
  $\{000, 001, 010, 011, 100, 101, 110, 111\}$
  where $b_2 b_1 b_0$ denotes state
  $\texttt{dreq} = b_2 \ \wedge\ \texttt{q0} = b_1 \ \wedge\ \texttt{dack} = b_0$
- Graph of the transition relation:



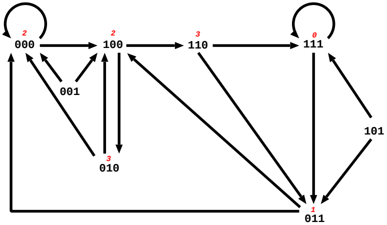
Computing Reachable M_{RCV}



- Define:

$$
\begin{aligned}
\mathcal{S}_0 &= \{b_2 b_1 b_0 \mid b_2 b_1 b_0 \in \{111\}\} \\
&= \{111\} \\
\mathcal{S}_{i+1} &= \mathcal{S}_i \ \cup \ \{s' \mid \exists s \in \mathcal{S}_i.\ R_{\mathrm{RCV}}\ s\ s'\ \} \\
&= \mathcal{S}_i \ \cup \ \{b_2' b_1' b_0' \mid \\
&\qquad \exists b_2 b_1 b_0 \in \mathcal{S}_i.\ (b_1' = b_2)\ \wedge\ (b_0' = b_2 \wedge (b_1 \vee b_0))\}
\end{aligned}
$$

Computing Reachable M_{RCV} (continued)



- Compute:

$$
\begin{aligned}
\mathcal{S}_0 &= \{111\} \\
\mathcal{S}_1 &= \{111\} \cup \{011\} \\
&= \{111, 011\} \\
\mathcal{S}_2 &= \{111, 011\} \cup \{000, 100\} \\
&= \{111, 011, 000, 100\} \\
\mathcal{S}_3 &= \{111, 011, 000, 100\} \cup \{010, 110\} \\
&= \{111, 011, 000, 100, 010, 110\} \\
\mathcal{S}_i &= \mathcal{S}_3 \quad (i > 3)
\end{aligned}
$$

- Hence Reachable $M_{\mathrm{RCV}} = \{111, 011, 000, 100, 010, 110\}$

Model checking $M_{\mathrm{RCV}} \models \mathbf{AG}\, p$

- $M = (S_{\mathrm{RCV}}, \{111\}, R_{\mathrm{RCV}}, L_{\mathrm{RCV}})$
- To check $M_{\mathrm{RCV}} \models \mathbf{AG}\, p$
  - compute Reachable $M_{\mathrm{RCV}} = \{111, 011, 000, 100, 010, 110\}$
  - check Reachable $M_{\mathrm{RCV}} \subseteq \{s \mid p \in L_{\mathrm{RCV}}(s)\}$, i.e. check:

    $p \in L_{\mathrm{RCV}}(111)$
    $p \in L_{\mathrm{RCV}}(011)$
    $p \in L_{\mathrm{RCV}}(000)$
    $p \in L_{\mathrm{RCV}}(100)$
    $p \in L_{\mathrm{RCV}}(010)$
    $p \in L_{\mathrm{RCV}}(110)$

Symbolic Boolean model checking of reachability

- Assume states are n-tuples of Booleans $(b_1, \ldots, b_n)$
  - $b_i \in \mathbb{B} = \{true, false\}$
  - $S = \mathbb{B}^n$, so S is finite: 2^n states
- Assume n distinct Boolean variables: $v_1, \ldots, v_n$
  - e.g. if $n = 3$ then could have $v_1 = \mathtt{x}$, $v_2 = \mathtt{y}$, $v_3 = \mathtt{z}$
- Boolean formula $f(v_1, \ldots, v_n)$ represents a subset of S
  - $f(v_1, \ldots, v_n)$ only contains variables $v_1, \ldots, v_n$
  - $f(b_1, \ldots, b_n)$ denotes result of substituting b_i for v_i
  - $f(v_1, \ldots, v_n)$ determines $\{(b_1, \ldots, b_n) | f(b_1, \ldots, b_n) \Leftrightarrow true\}$
- Example $\neg(\mathtt{x} = \mathtt{y})$ represents $\{(true, false), (false, true)\}$
- Transition relations also represented by Boolean formulae
  - e.g. $R_{\mathtt{RCV}}$ represented by:
    $(q0' = dreq) \wedge (dack' = (dreq \wedge (q0 \vee (\neg q0 \wedge dack))))$

Symbolically represent Boolean formulae as BDDs

- Key features of Binary Decision Diagrams (BDDs):
  - canonical (given a variable ordering)
  - efficient to manipulate
- Variables:

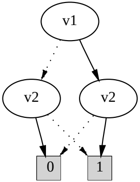
```
v   =  if v then 1 else 0
¬v  =  if v then 0 else 1
```

- Example: BDDs of variable `v` and `¬v`

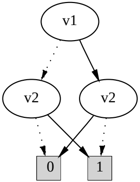
- Example: BDDs of $v1 \wedge v2$ and $v1 \vee v2$

More BDD examples

- BDD of $\texttt{v1} = \texttt{v2}$



- BDD of $\texttt{v1} \neq \texttt{v2}$

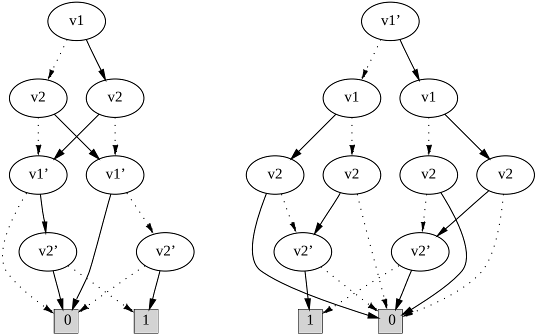


BDD of a transition relation

- BDDs of

$$(\mathtt{v1}' = (\mathtt{v1} = \mathtt{v2})) \wedge (\mathtt{v2}' = (\mathtt{v1} \neq \mathtt{v2}))$$

  with two different variable orderings



- **Exercise:** draw BDD of R_{RCV}

Standard BDD operations

- If formulae f_1, f_2 represents sets S_1, S_2, respectively then $f_1 \wedge f_2$, $f_1 \vee f_2$ represent $S_1 \cap S_2$, $S_1 \cup S_2$ respectively
- Standard algorithms compute Boolean operation on BDDs
- Abbreviate $(v_1, \ldots, v_n)$ to $\vec{v}$
- If $f(\vec{v})$ represents S
  and $g(\vec{v}, \vec{v}')$ represents $\{(\vec{v}, \vec{v}') \mid R\ \vec{v}\ \vec{v}'\}$
  then $\exists \vec{u}.\ f(\vec{u}) \wedge g(\vec{u}, \vec{v})$ represents $\{\vec{v} \mid \exists \vec{u}.\ \vec{u} \in S \wedge R\ \vec{u}\ \vec{v}\}$
- Can compute BDD of $\exists \vec{u}.\ h(\vec{u}, \vec{v})$ from BDD of $h(\vec{u}, \vec{v})$
  - e.g. BDD of $\exists v_1.\ h(v_1, v_2)$ is BDD of $h(\mathtt{T}, v_2) \vee h(\mathtt{F}, v_2)$
- From BDD of formula $f(v_1, \ldots, v_n)$ can compute b_1, ..., b_n such that if $v_1 = b_1$, ..., $v_n = b_n$ then $f(b_1, \ldots, b_n) \Leftrightarrow \mathit{true}$
  - b_1, ..., b_n is a satisfying assignment (SAT problem)
  - used for counterexample generation (see later)

Reachable States via BDDs

- Assume $M = (S, S_0, R, L)$ and $S = \mathbb{B}^n$
- Represent R by Boolean formulae $g(\vec{v}, \vec{v'})$
- Iteratively define formula $f_n(\vec{v})$ representing $\mathcal{S}_n$

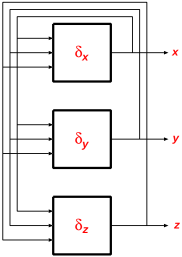
$$
\begin{aligned}
f_0(\vec{v}) \quad &= \text{formula representing } S_0 \\
f_{n+1}(\vec{v}) \quad &= f_n(\vec{v}) \vee (\exists \vec{u}.\ f_n(\vec{u}) \wedge g(\vec{u}, \vec{v}))
\end{aligned}
$$

- Let $\mathcal{B}_0$, $\mathcal{B}_R$ be BDDs representing $f_0(\vec{v})$, $g(\vec{v}, \vec{v'})$
- Iteratively compute BDDs $\mathcal{B}_n$ representing f_n

$$\mathcal{B}_{n+1} = \mathcal{B}_n \ \underline{\vee} \ (\underline{\exists \vec{u}.} \ \mathcal{B}_n\underline{[\vec{u}/\vec{v}]} \ \underline{\wedge} \ \mathcal{B}_R\underline{[\vec{u}, \vec{v}/\vec{v}, \vec{v'}]})$$

  - efficient using (blue underlined) standard BDD algorithms (renaming, conjunction, disjunction, quantification)
  - BDD $\mathcal{B}_n$ only contains variables $\vec{v}$: represents $\mathcal{S}_n \subseteq S$
- At each iteration check $\mathcal{B}_{n+1} = \mathcal{B}_n$ efficient using BDDs
  - when $\mathcal{B}_{n+1} = \mathcal{B}_n$ can conclude $\mathcal{B}_n$ represents Reachable M
  - we call this BDD $\mathcal{B}_M$ in a later slide (i.e. $\mathcal{B}_M = \mathcal{B}_n$)

Example BDD optimisation: disjunctive partitioning



Three state transition functions in parallel

$$\delta_x, \delta_y, \delta_z : \mathbb{B} \times \mathbb{B} \times \mathbb{B} \to \mathbb{B}$$

- Transition relation (asynchronous interleaving semantics):

$$\begin{aligned}
&R\,(x, y, z)\,(x', y', z') = \\
&\quad (x' = \delta_x(x, y, z) \;\wedge\; y' = y \;\wedge\; z' = z) \vee \\
&\quad (x' = x \;\wedge\; y' = \delta_y(x, y, z) \;\wedge\; z' = z) \vee \\
&\quad (x' = x \;\wedge\; y' = y \;\wedge\; z' = \delta_z(x, y, z))
\end{aligned}$$

Avoiding building big BDDs

- Transition relation for three transition functions in parallel

$$
\begin{array}{l}
R(x,y,z)\ (x',y',z') = \\
\quad (x' = \delta_x(x,y,z) \ \wedge\ y' = y \ \wedge\ z' = z) \vee \\
\quad (x' = x \ \wedge\ y' = \delta_y(x,y,z) \ \wedge\ z' = z) \vee \\
\quad (x' = x \ \wedge\ y' = y \ \wedge\ z' = \delta_z(x,y,z))
\end{array}
$$

- Recall symbolic iteration:

$$f_{n+1}(\vec{v}) \ = f_n(\vec{v}) \vee (\exists \vec{u}.\ f_n(\vec{u}) \wedge g(\vec{u},\vec{v}))$$

- For this particular R (see next slide):

$$
\begin{array}{l}
f_{n+1}(x,y,z) \\
\ = f_n(x,y,z) \vee (\exists \overline{x}\ \overline{y}\ \overline{z}.\ f_n(\overline{x},\overline{y},\overline{z}) \wedge R\ (\overline{x},\overline{y},\overline{z})\ (x,y,z)) \\
\ = f_n(x,y,z) \hfill \vee \\
\qquad (\exists \overline{x}.\ f_n(\overline{x},y,z) \ \wedge\ x = \delta_x(\overline{x},y,z)) \vee \\
\qquad (\exists \overline{y}.\ f_n(x,\overline{y},z) \ \wedge\ y = \delta_y(x,\overline{y},z)) \vee \\
\qquad (\exists \overline{z}.\ f_n(x,y,\overline{z}) \ \wedge\ z = \delta_z(x,y,\overline{z}))
\end{array}
$$

- Don't need to calculate BDD of R!

Disjunctive partitioning – Exercise: understand this

$$
\begin{array}{l}
\exists\overline{x}\ \overline{y}\ \overline{z}.\ f_n(\overline{x},\overline{y},\overline{z}) \ \wedge\ R\ (\overline{x},\overline{y},\overline{z})\ (x,y,z) \\
\quad = \ \exists\overline{x}\ \overline{y}\ \overline{z}.\ f_n(\overline{x},\overline{y},\overline{z}) \ \wedge\ ((x = \delta_x(\overline{x},\overline{y},\overline{z}) \ \wedge\ y = \overline{y} \ \wedge\ z = \overline{z}) \vee \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad (x = \overline{x} \ \wedge\ y = \delta_y(\overline{x},\overline{y},\overline{z}) \ \wedge\ z = \overline{z}) \vee \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad (x = \overline{x} \ \wedge\ y = \overline{y} \ \wedge\ z = \delta_z(\overline{x},\overline{y},\overline{z})))
\end{array}
$$

$$
\begin{array}{rl}
= & (\exists\overline{x}\ \overline{y}\ \overline{z}.\ f_n(\overline{x},\overline{y},\overline{z}) \ \wedge\ x = \delta_x(\overline{x},\overline{y},\overline{z}) \ \wedge\ y = \overline{y} \ \wedge\ z = \overline{z}) \vee \\
 & (\exists\overline{x}\ \overline{y}\ \overline{z}.\ f_n(\overline{x},\overline{y},\overline{z}) \ \wedge\ x = \overline{x} \ \wedge\ y = \delta_y(\overline{x},\overline{y},\overline{z}) \ \wedge\ z = \overline{z}) \vee \\
 & (\exists\overline{x}\ \overline{y}\ \overline{z}.\ f_n(\overline{x},\overline{y},\overline{z}) \ \wedge\ x = \overline{x} \ \wedge\ y = \overline{y} \ \wedge\ z = \delta_z(\overline{x},\overline{y},\overline{z}))
\end{array}
$$

$$
\begin{array}{rl}
= & (\exists\overline{x}\ \overline{y}\ \overline{z}.\ f_n(\overline{x},y,z) \ \wedge\ x = \delta_x(\overline{x},y,z) \ \wedge\ y = \overline{y} \ \wedge\ z = \overline{z}) \vee \\
 & (\exists\overline{x}\ \overline{y}\ \overline{z}.\ f_n(x,\overline{y},z) \ \wedge\ x = \overline{x} \ \wedge\ y = \delta_y(x,\overline{y},z) \ \wedge\ z = \overline{z}) \vee \\
 & (\exists\overline{x}\ \overline{y}\ \overline{z}.\ f_n(x,y,\overline{z}) \ \wedge\ x = \overline{x} \ \wedge\ y = \overline{y} \ \wedge\ z = \delta_z(x,y,\overline{z}))
\end{array}
$$

$$
\begin{array}{rl}
= & ((\exists\overline{x}.\ f_n(\overline{x},y,z) \wedge x{=}\delta_x(\overline{x},y,z)) \ \wedge\ (\exists\overline{y}.\ y{=}\overline{y}) \ \wedge\ (\exists\overline{z}.\ z{=}\overline{z})) \vee \\
 & ((\exists\overline{x}.\ x{=}\overline{x}) \ \wedge\ (\exists\overline{y}.\ f_n(x,\overline{y},z) \wedge y{=}\delta_y(x,\overline{y},z)) \ \wedge\ (\exists\overline{z}.\ z{=}\overline{z})) \vee \\
 & ((\exists\overline{x}.\ x{=}\overline{x}) \ \wedge\ (\exists\overline{y}.\ y{=}\overline{y}) \ \wedge\ (\exists\overline{z}.\ f_n(x,y,\overline{z}) \wedge z{=}\delta_z(x,y,\overline{z})))
\end{array}
$$

$$
\begin{array}{rl}
= & (\exists\overline{x}.\ f_n(\overline{x},y,z) \ \wedge\ x = \delta_x(\overline{x},y,z)) \vee \\
 & (\exists\overline{y}.\ f_n(x,\overline{y},z) \ \wedge\ y = \delta_y(x,\overline{y},z)) \vee \\
 & (\exists\overline{z}.\ f_n(x,y,\overline{z}) \ \wedge\ z = \delta_z(x,y,\overline{z}))
\end{array}
$$

Verification and counterexamples

- Typical safety question:
  - is property p true in all reachable states?
  - i.e. check $M \models \mathbf{AG}\, p$
  - i.e. is $\forall s.\ s \in \mathrm{Reachable}\ M \Rightarrow p\ s$
- Check using BDDs
  - compute BDD $\mathcal{B}_M$ of $\mathrm{Reachable}\ M$
  - compute BDD $\mathcal{B}_p$ of $p(\vec{v})$
  - check if BDD of $\mathcal{B}_M \underline{\Rightarrow} \mathcal{B}_p$ is the single node $\boxed{1}$
- Valid because *true* represented by a unique BDD (canonical property)
- If BDD is not $\boxed{1}$ can get counterexample

Generating counterexamples (general idea)

BDD algorithms can find satisfying assignments (SAT)

- Suppose not all reachable states of model M satisfy p
- i.e. $\exists s \in \mathrm{Reachable}\ M.\ \neg(p(s))$
- Set of reachable state $\mathcal{S}$ given by: $\mathcal{S} = \bigcup_{n=0}^{\infty} \mathcal{S}_n$
- Iterate to find least n such that $\exists s \in \mathcal{S}_n.\ \neg(p(s))$
- Use SAT to find b_n such that $b_n \in \mathcal{S}_n \wedge \neg(p(b_n))$
- Use SAT to find b_{n-1} such that $b_{n-1} \in \mathcal{S}_{n-1} \wedge R\ b_{n-1}\ b_n$
- Use SAT to find b_{n-2} such that $b_{n-2} \in \mathcal{S}_{n-2} \wedge R\ b_{n-2}\ b_{n-1}$

  $\vdots$
- Iterate to find b_0, b_1, ..., b_{n-1}, b_n where $b_i \in \mathcal{S}_i \wedge R\ b_{i-1\ i}$
- Then $b_0\ b_1\ \cdots\ b_{n-1}\ b_n$ is a path to a counterexample

Use SAT to find s_{n-1} such that $s_{n-1} \in \mathcal{S}_{n-1} \wedge R\ s_{n-1}\ s_n$

- Suppose states s, s' symbolically represented by $\vec{v}$, $\vec{v'}$
- Suppose BDD $\mathcal{B}_i$ represents $\vec{v} \in \mathcal{S}_i$ $(1 \leq i \leq n)$
- Suppose BDD $\mathcal{B}_R$ represents $R\ \vec{v}\ \vec{v'}$
- Then BDD
  $(\mathcal{B}_{n-1}\ \underline{\wedge}\ \mathcal{B}_R\underline{[\vec{b}_n/\vec{v'}]})$
  represents
  $\vec{v} \in \mathcal{S}_{n-1} \wedge R\ \vec{v}\ \vec{b}_n$
- Use SAT to find a valuation $\vec{b}_{n-1}$ for $\vec{v}$
- Then BDD
  $(\mathcal{B}_{n-1}\ \underline{\wedge}\ \mathcal{B}_R\underline{[\vec{b}_n/\vec{v'}]})\underline{[\vec{b}_{n-1}/\vec{v}]}$
  represents
  $\vec{b}_{n-1} \in \mathcal{S}_{n-1} \wedge R\ \vec{b}_{n-1}\ \vec{b}_n \quad = \quad \boxed{1}$

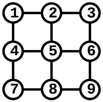
Generating counterexamples with BDDs

BDD algorithms can find satisfying assignments (SAT)

- $M = (S, S_0, R, L)$ and $\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_M, \mathcal{B}_R, \mathcal{B}_p$ as earlier
- Suppose $\mathcal{B}_M \underline{\Rightarrow} \mathcal{B}_p$ is not $\boxed{1}$
- Must exist a state $s \in \text{Reachable } M$ such that $\neg(p\ s)$
- Let $\mathcal{B}_{\neg p}$ be the BDD representing $\neg(p\ \vec{v})$
- Iterate to find first n such that $\mathcal{B}_n \underline{\wedge} \mathcal{B}_{\neg p}$
- Use SAT to find $\vec{b}_n$ such that $(\mathcal{B}_n \underline{\wedge} \mathcal{B}_{\neg p})\underline{[\vec{b}_n/\vec{v}]}$
- Use SAT to find $\vec{b}_{n-1}$ such that $(\mathcal{B}_{n-1} \underline{\wedge} \mathcal{B}_R\underline{[\vec{b}_n/\vec{v}']})\underline{[\vec{b}_{n-1}/\vec{v}]}$
- For $0 < i < n$ find $\vec{b}_{i-1}$ such that $(\mathcal{B}_{i-1} \underline{\wedge} \mathcal{B}_R\underline{[\vec{b}_i/\vec{v}']})\underline{[\vec{b}_{i-1}/\vec{v}]}$
- $\vec{b}_0, \ldots, \vec{b}_i, \ldots, \vec{b}_n$ is a counterexample trace
- Sometimes can use partitioning to avoid constructing $\mathcal{B}_R$

Example (from an exam)

Consider a 3x3 array of 9 switches



Suppose each switch 1,2,...,9 can either be on or off, and that toggling any switch will automatically toggle all its immediate neighbours. For example, toggling switch 5 will also toggle switches 2, 4, 6 and 8, and toggling switch 6 will also toggle switches 3, 5 and 9.

(a) Devise a state space [4 marks] and transition relation [6 marks] to represent the behavior of the array of switches

You are given the problem of getting from an initial state in which even numbered switches are on and odd numbered switches are off, to a final state in which all the switches are off.

(b) Write down predicates on your state space that characterises the initial [2 marks] and final [2 marks] states.

(c) Explain how you might use a model checker to find a sequences of switches to toggle to get from the initial to final state. [6 marks]

You are not expected to actually solve the problem, but only to explain how to represent it in terms of model checking.

Solution

A state is a vector $(v1, v2, v3, v4, v5, v6, v7, v8, v9)$, where $vi \in \mathbb{B}$

A transition relation `Trans` is then defined by:

$$
\begin{array}{ll}
\mathtt{Trans}(v1,v2,v3,v4,v5,v6,v7,v8,v9)(v1',v2',v3',v4',v5',v6',v7',v8',v9') & \\
\ = ((v1'=\neg v1)\land(v2'=\neg v2)\land(v3'=v3)\land(v4'=\neg v4)\land(v5'=v5)\land & \\
\quad\ (v6'=v6)\land(v7'=v7)\land(v8'=v8)\land(v9'=v9)) & \text{(toggle switch 1)} \\
\ \lor\ ((v1'=\neg v1)\land(v2'=\neg v2)\land(v3'=\neg v3)\land(v4'=v4)\land(v5'=\neg v5)\land & \\
\quad\ (v6'=v6)\land(v7'=v7)\land(v8'=v8)\land(v9'=v9)) & \text{(toggle switch 2)} \\
\ \lor\ ((v1'=v1)\land(v2'=\neg v2)\land(v3'=\neg v3)\land(v4'=v4)\land(v5'=v5)\land & \\
\quad\ (v6'=\neg v6)\land(v7'=v7)\land(v8'=v8)\land(v9'=v9)) & \text{(toggle switch 3)} \\
\ \lor\ ((v1'=\neg v1)\land(v2'=v2)\land(v3'=v3)\land(v4'=\neg v4)\land(v5'=\neg v5)\land & \\
\quad\ (v6'=v6)\land(v7'=\neg v7)\land(v8'=v8)\land(v9'=v9)) & \text{(toggle switch 4)} \\
\ \lor\ ((v1'=v1)\land(v2'=\neg v2)\land(v3'=v3)\land(v4'=\neg v4)\land(v5'=\neg v5)\land & \\
\quad\ (v6'=\neg v6)\land(v7'=v7)\land(v8'=\neg v8)\land(v9'=v9)) & \text{(toggle switch 5)} \\
\ \lor\ ((v1'=v1)\land(v2'=v2)\land(v3'=\neg v3)\land(v4'=v4)\land(v5'=\neg v5)\land & \\
\quad\ (v6'=\neg v6)\land(v7'=v7)\land(v8'=v8)\land(v9'=\neg v9)) & \text{(toggle switch 6)} \\
\ \lor\ ((v1'=v1)\land(v2'=v2)\land(v3'=v3)\land(v4'=\neg v4)\land(v5'=v5)\land & \\
\quad\ (v6'=v6)\land(v7'=\neg v7)\land(v8'=\neg v8)\land(v9'=v9)) & \text{(toggle switch 7)} \\
\ \lor\ ((v1'=v1)\land(v2'=v2)\land(v3'=v3)\land(v4'=v4)\land(v5'=\neg v5)\land & \\
\quad\ (v6'=v6)\land(v7'=\neg v7)\land(v8'=\neg v8)\land(v9'=\neg v9)) & \text{(toggle switch 8)} \\
\ \lor\ ((v1'=v1)\land(v2'=v2)\land(v3'=v3)\land(v4'=v4)\land(v5'=v5)\land & \\
\quad\ (v6'=\neg v6)\land(v7'=v7)\land(v8'=\neg v8)\land(v9'=\neg v9)) & \text{(toggle switch 9)}
\end{array}
$$

Solution (continued)

Predicates `Init`, `Final` characterising the initial and final states, respectively, are defined by:

```
Init(v1,v2,v3,v4,v5,v6,v7,v8,v9) =
 ¬v1 ∧ v2 ∧ ¬v3 ∧ v4 ∧ ¬v5 ∧ v6 ∧ ¬v7 ∧ v8 ∧ ¬v9

Final(v1,v2,v3,v4,v5,v6,v7,v8,v9) =
 ¬v1 ∧ ¬v2 ∧ ¬v3 ∧ ¬v4 ∧ ¬v5 ∧ ¬v6 ∧ ¬v7 ∧ ¬v8 ∧ ¬v9
```

Model checkers can find counter-examples to properties, and sequences of transitions from an initial state to a counter-example state. Thus we could use a model checker to find a trace to a counter-example to the property that

```
¬Final(v1,v2,v3,v4,v5,v6,v7,v8,v9)
```

Properties

- $\forall s{\in}S_0.\forall s'.R^*\ s\ s' \Rightarrow p\ s'$ says p true in all reachable states
- Might want to verify other properties
  1. `DeviceEnabled` holds infinitely often along every path
  2. From any state it is possible to get to a state where `Restart` holds
  3. After a three or more consecutive occurrences of `Req` there will eventually be an `Ack`
- Temporal logic can express such properties
- There are several temporal logics in use
  - LTL is good for the first example above
  - CTL is good for the second example
  - PSL is good for the third example
- Model checking:
  - Emerson, Clarke & Sifakis: Turing Award 2008
  - widely used in industry: first hardware, later software

Temporal logic (originally called “tense logic”)

A. N. Prior
1914-1969

Originally devised for investigating: “the relationship between tense and modality attributed to the Megarian philosopher Diodorus Cronus (ca. 340-280 BCE)”.

Mary Prior, his wife, recalls “I remember his waking me one night [in 1953], coming and sitting on my bed, ... and saying he thought one could make a formalised tense logic”.

- Temporal logic: deductive system for reasoning about time
  - temporal formulae for expressing temporal statements
  - deductive system for proving theorems
- Temporal logic model checking
  - uses semantics to check truth of temporal formulae in models
- Temporal logic proof systems also important in CS
  - use pioneered by Amir Pnueli (1996 Turing Award)
  - not considered in this course

Recommended: `http://plato.stanford.edu/entries/prior/`

Temporal logic formulae (statements)

- Many different languages of temporal statements
  - linear time (LTL)
  - branching time (CTL)
  - finite intervals (SEREs)
  - industrial languages (PSL, SVA)
- Prior used linear time, Kripke suggested branching time:

  ... we perhaps should not regard time as a linear series ... there are several possibilities for what the next moment may be like - and for each possible next moment, there are several possibilities for the moment after that. Thus the situation takes the form, not of a linear sequence, but of a 'tree'.

  [Saul Kripke, 1958 (aged 17, still at school)]
- CS issues different from philosophical issues
  - Moshe Vardi: "Branching vs. Linear Time: Final Showdown"

http://www.computer.org/portal/web/awards/Vardi

Moshe Vardi
www.computer.org

"For fundamental and lasting contributions to the development of logic as a unifying foundational framework and a tool for modeling computational systems"

2011 Harry H. Goode Memorial Award Recipient

Linear Temporal Logic (LTL)

- Grammar of *well formed formulae* (wff) ϕ

$$
\begin{array}{rcll}
\phi & ::= & p & \text{(Atomic formula: } p \in AP\text{)} \\
 & | & \neg\phi & \text{(Negation)} \\
 & | & \phi_1 \vee \phi_2 & \text{(Disjunction)} \\
 & | & \mathbf{X}\phi & \text{(successor)} \\
 & | & \mathbf{F}\phi & \text{(sometimes)} \\
 & | & \mathbf{G}\phi & \text{(always)} \\
 & | & [\phi_1\ \mathbf{U}\ \phi_2] & \text{(Until)}
\end{array}
$$

- Details differ from Prior's tense logic – but similar ideas

- Semantics define when ϕ true in model M
  - where $M = (S, R, S_0, L)$ – a Kripke structure
  - notation: $M \models \phi$ means ϕ true in model M
  - model checking algorithms compute this (when decidable)

$M \models \phi$ means "wff ϕ is true in model M"

- If $M = (S, S_0, R, L)$ then

  $\boxed{\pi \text{ is an } M\text{-path starting from } s \text{ iff Path } R\ s\ \pi}$

- If $M = (S, S_0, R, L)$ then we define $M \models \phi$ to mean:

  $\boxed{\phi \text{ is true on all } M\text{-paths starting from a member of } S_0}$

- We will define $[\![\phi]\!]_M(\pi)$ to mean

  $\boxed{\phi \text{ is true on the } M\text{-path } \pi}$

- Thus $M \models \phi$ will be formally defined by:

  $\boxed{M \models \phi \ \Leftrightarrow\ \forall \pi\ s.\ s \in S_0 \wedge \text{Path } R\ s\ \pi \Rightarrow [\![\phi]\!]_M(\pi)}$

- It remains to actually define $[\![\phi]\!]_M$ for all wffs ϕ

Definition of $[\![\phi]\!]_M(\pi)$

- $[\![\phi]\!]_M(\pi)$ is the application of function $[\![\phi]\!]_M$ to path π
  - thus $[\![\phi]\!]_M : (\mathbb{N} \to S) \to \mathbb{B}$
- Let $M = (S, S_0, R, L)$

  $[\![\phi]\!]_M$ is defined by structural induction on ϕ

$$
\begin{array}{lcl}
[\![p]\!]_M(\pi) & = & p \in L(\pi\ 0) \\
[\![\neg\phi]\!]_M(\pi) & = & \neg([\![\phi]\!]_M(\pi)) \\
[\![\phi_1 \vee \phi_2]\!]_M(\pi) & = & [\![\phi_1]\!]_M(\pi) \ \vee \ [\![\phi_2]\!]_M(\pi) \\
[\![\mathbf{X}\phi]\!]_M(\pi) & = & [\![\phi]\!]_M(\pi{\downarrow}1) \\
[\![\mathbf{F}\phi]\!]_M(\pi) & = & \exists i.\ [\![\phi]\!]_M(\pi{\downarrow}i) \\
[\![\mathbf{G}\phi]\!]_M(\pi) & = & \forall i.\ [\![\phi]\!]_M(\pi{\downarrow}i) \\
[\![[\phi_1\ \mathbf{U}\ \phi_2]]\!]_M(\pi) & = & \exists i.\ [\![\phi_2]\!]_M(\pi{\downarrow}i) \ \wedge \ \forall j.\ j{<}i \ \Rightarrow \ [\![\phi_1]\!]_M(\pi{\downarrow}j)
\end{array}
$$

- We look at each of these semantic equations in turn

$[\![p]\!]_M(\pi) = p(\pi\ 0)$

- Assume $M = (S, S_0, R, L)$
- We have: $[\![p]\!]_M(\pi) = p \in L(\pi\ 0)$
  - p is an atomic property, i.e. $p \in AP$
  - $\pi : \mathbb{N} \to S$ so $\pi\ 0 \in S$
  - $\pi\ 0$ is the first state in path π
  - $p \in L(\pi\ 0)$ is *true* iff atomic property p holds of state $\pi\ 0$
- $[\![p]\!]_M(\pi)$ means p holds of the first state in path π
- Assume $\mathtt{T}, \mathtt{F} \in AP$ with for all s: $\mathtt{T} \in L(s)$ and $\mathtt{F} \notin L(s)$
  - $[\![\mathtt{T}]\!]_M(\pi)$ is always true
  - $[\![\mathtt{F}]\!]_M(\pi)$ is always false

$$[\![\neg\phi]\!]_M(\pi) = \neg([\![\phi]\!]_M(\pi))$$
$$[\![\phi_1 \vee \phi_2]\!]_M(\pi) = [\![\phi_1]\!]_M(\pi) \vee [\![\phi_2]\!]_M(\pi)$$

- $[\![\neg\phi]\!]_M(\pi) = \neg([\![\phi]\!]_M(\pi))$
  - $[\![\neg\phi]\!]_M(\pi)$ true iff $[\![\phi]\!]_M(\pi)$ is not true

- $[\![\phi_1 \vee \phi_2]\!]_M(\pi) = [\![\phi_1]\!]_M(\pi) \vee [\![\phi_2]\!]_M(\pi)$
  - $[\![\phi_1 \vee \phi_2]\!]_M(\pi)$ true iff $[\![\phi_1]\!]_M(\pi)$ is true or $[\![\phi_2]\!]_M(\pi)$ is true

$[\![\mathbf{X}\phi]\!]_M(\pi) = [\![\phi]\!]_M(\pi{\downarrow}1)$

- $[\![\mathbf{X}\phi]\!]_M(\pi) = [\![\phi]\!]_M(\pi{\downarrow}1)$
  - $\pi{\downarrow}1$ is π with the first state chopped off

$$\begin{array}{l} \pi{\downarrow}1(0) = \pi(1+0) = \pi(1) \\ \pi{\downarrow}1(1) = \pi(1+1) = \pi(2) \\ \pi{\downarrow}1(2) = \pi(1+2) = \pi(3) \\ \qquad\qquad \vdots \end{array}$$

- $[\![\mathbf{X}\phi]\!]_M(\pi)$ true iff $[\![\phi]\!]_M$ true *starting at the next state of* π

$[\![\mathbf{F}\phi]\!]_M(\pi) = \exists i.\ [\![\phi]\!]_M(\pi{\downarrow}i)$

- $[\![\mathbf{F}\phi]\!]_M(\pi) = \exists i.\ [\![\phi]\!]_M(\pi{\downarrow}i)$
  - $\pi{\downarrow}i$ is π with the first i states chopped off

    $$\begin{aligned} \pi{\downarrow}i(0) &= \pi(i+0) = \pi(i) \\ \pi{\downarrow}i(1) &= \pi(i+1) \\ \pi{\downarrow}i(2) &= \pi(i+2) \\ &\vdots \end{aligned}$$

  - $[\![\phi]\!]_M(\pi{\downarrow}i)$ true iff $[\![\phi]\!]_M$ true *starting i states along* π

- $[\![\mathbf{F}\phi]\!]_M(\pi)$ true iff $[\![\phi]\!]_M$ true *starting somewhere along* π

- "$\mathbf{F}\phi$" is read as "sometimes ϕ"

$[\![\mathbf{G}\phi]\!]_M(\pi) = \forall i.\ [\![\phi]\!]_M(\pi{\downarrow}i)$

- $[\![\mathbf{G}\phi]\!]_M(\pi) = \forall i.\ [\![\phi]\!]_M(\pi{\downarrow}i)$
  - $\pi{\downarrow}i$ is π with the first i states chopped off
  - $[\![\phi]\!]_M(\pi{\downarrow}i)$ true iff $[\![\phi]\!]_M$ true *starting i states along* π
- $[\![\mathbf{G}\phi]\!]_M(\pi)$ true iff $[\![\phi]\!]_M$ true *starting anywhere along* π
- "$\mathbf{G}\phi$" is read as "always ϕ" or "globally ϕ"
- $M \models \mathbf{AG}\,p$ defined earlier: $M \models \mathbf{AG}\,p \Leftrightarrow M \models \mathbf{G}(p)$
- $\mathbf{G}$ is definable in terms of $\mathbf{F}$ and $\neg$: $\mathbf{G}\phi = \neg(\mathbf{F}(\neg\phi))$

$$\begin{array}{rcl}
[\![\neg(\mathbf{F}(\neg\phi))]\!]_M(\pi) & = & \neg([\![\mathbf{F}(\neg\phi)]\!]_M(\pi)) \\
& = & \neg(\exists i.\ [\![\neg\phi]\!]_M(\pi{\downarrow}i)) \\
& = & \neg(\exists i.\ \neg([\![\phi]\!]_M(\pi{\downarrow}i))) \\
& = & \forall i.\ [\![\phi]\!]_M(\pi{\downarrow}i) \\
& = & [\![\mathbf{G}\phi]\!]_M(\pi)
\end{array}$$

$$\boxed{[\![[\phi_1 \ \mathbf{U} \ \phi_2]]\!]_M(\pi) = \ \exists i.\ [\![\phi_2]\!]_M(\pi{\downarrow}i) \wedge \forall j.\ j{<}i \Rightarrow [\![\phi_1]\!]_M(\pi{\downarrow}j)}$$

- $[\![[\phi_1 \ \mathbf{U} \ \phi_2]]\!]_M(\pi) = \exists i.\ [\![\phi_2]\!]_M(\pi{\downarrow}i) \wedge \forall j.\ j{<}i \Rightarrow [\![\phi_1]\!]_M(\pi{\downarrow}j)$
  - $[\![\phi_2]\!]_M(\pi{\downarrow}i)$ true iff $[\![\phi_2]\!]_M$ true *starting i states along* π
  - $[\![\phi_1]\!]_M(\pi{\downarrow}j)$ true iff $[\![\phi_1]\!]_M$ true *starting j states along* π
- $[\![[\phi_1 \ \mathbf{U} \ \phi_2]]\!]_M(\pi)$ is true iff
  $[\![\phi_2]\!]_M$ is true somewhere along π and up to then $[\![\phi_1]\!]_M$ is true
- "$[\phi_1 \ \mathbf{U} \ \phi_2]$" is read as "$\phi_1$ until ϕ_2"
- **F** is definable in terms of $[- \ \mathbf{U} \ -]$: $\mathbf{F}\phi = [\mathtt{T} \ \mathbf{U} \ \phi]$

$$\begin{array}{l}
[\![[\mathtt{T} \ \mathbf{U} \ \phi]]\!]_M(\pi) \\
= \ \exists i.\ [\![\phi]\!]_M(\pi{\downarrow}i) \wedge \forall j.\ j{<}i \Rightarrow [\![\mathtt{T}]\!]_M(\pi{\downarrow}j) \\
= \ \exists i.\ [\![\phi]\!]_M(\pi{\downarrow}i) \wedge \forall j.\ j{<}i \Rightarrow \mathit{true} \\
= \ \exists i.\ [\![\phi]\!]_M(\pi{\downarrow}i) \wedge \mathit{true} \\
= \ \exists i.\ [\![\phi]\!]_M(\pi{\downarrow}i) \\
= \ [\![\mathbf{F}\phi]\!]_M(\pi)
\end{array}$$

Review of Linear Temporal Logic (LTL)

- Grammar of *well formed formulae* (wff) ϕ

$$
\begin{array}{lcll}
\phi & ::= & p & \text{(Atomic formula: } p \in AP\text{)} \\
 & | & \neg\phi & \text{(Negation)} \\
 & | & \phi_1 \vee \phi_2 & \text{(Disjunction)} \\
 & | & \mathbf{X}\phi & \text{(successor)} \\
 & | & \mathbf{F}\phi & \text{(sometimes)} \\
 & | & \mathbf{G}\phi & \text{(always)} \\
 & | & [\phi_1 \ \mathbf{U} \ \phi_2] & \text{(Until)}
\end{array}
$$

- $M \models \phi$ means ϕ holds on all M-paths
  - $M = (S, S_0, R, L)$
  - $[\![\phi]\!]_M(\pi)$ means ϕ is true on the M-path π
  - $M \models \phi \ \Leftrightarrow \ \forall\pi\ s.\ s \in S_0 \wedge \mathrm{Path}\ R\ s\ \pi \Rightarrow [\![\phi]\!]_M(\pi)$

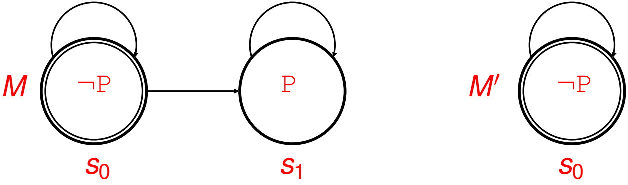
LTL examples

- "DeviceEnabled holds infinitely often along every path"
  $\boxed{\mathbf{G}(\mathbf{F}\ \texttt{DeviceEnabled})}$
- "Eventually the state becomes permanently Done"
  $\boxed{\mathbf{F}(\mathbf{G}\ \texttt{Done})}$
- "Every Req is followed by an Ack"
  $\boxed{\mathbf{G}(\texttt{Req} \Rightarrow \mathbf{F}\ \texttt{Ack})}$
  Number of Req and Ack may differ - no counting
- "If Enabled infinitely often then Running infinitely often"
  $\boxed{\mathbf{G}(\mathbf{F}\ \texttt{Enabled}) \Rightarrow \mathbf{G}(\mathbf{F}\ \texttt{Running})}$
- "An upward going lift at the second floor keeps going up if a passenger requests the fifth floor"
  $$\boxed{\begin{array}{l}\mathbf{G}(\texttt{AtFloor2} \land \texttt{DirectionUp} \land \texttt{RequestFloor5} \\ \quad \Rightarrow [\texttt{DirectionUp}\ \mathbf{U}\ \texttt{AtFloor5}])\end{array}}$$

(acknowledgement: http://pswlab.kaist.ac.kr/courses/cs402-2011/temporal-logic2.pdf)

A property not expressible in LTL

- Consider models M and M' below



$$
\begin{array}{lcl}
M & = & (\{s_0, s_1\}, \{s_0\}, \{(s_0, s_0), (s_0, s_1), (s_1, s_1)\}, L) \\
M' & = & (\{s_0\}, \{s_0\}, \{(s_0, s_0)\}, L)
\end{array}
$$

where: $L = \lambda s.\ \mathit{if}\ s = s_0\ \mathit{then}\ \{\}\ \mathit{else}\ \{\mathtt{P}\}$

- Every M'-path is also an M-path
- So if ϕ true on every M-path then ϕ true on every M'-path
- Hence for any ϕ if $M \models \phi$ then $M' \models \phi$
- Consider property "can always reach a state satisfying $\mathtt{P}$"
  - true: $M \models$ "can always reach a state satisfying $\mathtt{P}$"
  - false: $M' \not\models$ "can always reach a state satisfying $\mathtt{P}$"
- "can always reach a state satisfying $\mathtt{P}$" not expressible in LTL

LTL expressibility

"can always reach a state satisfying `p`"

- In LTL $M \models \phi$ says ϕ holds of all paths of M
- LTL formulae ϕ are evaluated on paths .... path formulae
- Want to say there exists a path to $p \in AP$
  - $\exists \pi.\ \mathrm{Path}\ R\ s\ \pi\ \wedge\ \exists i.\ p \in L(\pi(i))$
- CTL properties are evaluated at a state ... state formulae
  - they can talk about both some or all paths
  - starting from the state they are evaluated at

Computation Tree Logic (CTL)

- LTL formulae ϕ are evaluated on paths .... path formulae
- CTL formulae ψ are evaluated on states .. state formulae

- Syntax of CTL well-formed formulae:

$$
\begin{array}{rcll}
\psi & ::= & p & \text{(Atomic formula } p \in AP\text{)} \\
 & | & \neg\psi & \text{(Negation)} \\
 & | & \psi_1 \wedge \psi_2 & \text{(Conjunction)} \\
 & | & \psi_1 \vee \psi_2 & \text{(Disjunction)} \\
 & | & \psi_1 \Rightarrow \psi_2 & \text{(Implication)} \\
 & | & \mathbf{AX}\psi & \text{(All successors)} \\
 & | & \mathbf{EX}\psi & \text{(Some successors)} \\
 & | & \mathbf{A}[\psi_1\ \mathbf{U}\ \psi_2] & \text{(Until – along all paths)} \\
 & | & \mathbf{E}[\psi_1\ \mathbf{U}\ \psi_2] & \text{(Until – along some path)}
\end{array}
$$

Semantics of CTL

- Assume $M = (S, S_0, R, L)$ and then define:

$$
\begin{array}{lcl}
[\![p]\!]_M(s) & = & p \in L(s) \\
[\![\neg\psi]\!]_M(s) & = & \neg([\![\psi]\!]_M(s)) \\
[\![\psi_1 \wedge \psi_2]\!]_M(s) & = & [\![\psi_1]\!]_M(s) \;\wedge\; [\![\psi_2]\!]_M(s) \\
[\![\psi_1 \vee \psi_2]\!]_M(s) & = & [\![\psi_1]\!]_M(s) \;\vee\; [\![\psi_2]\!]_M(s) \\
[\![\psi_1 \Rightarrow \psi_2]\!]_M(s) & = & [\![\psi_1]\!]_M(s) \;\Rightarrow\; [\![\psi_2]\!]_M(s) \\
[\![\mathbf{AX}\psi]\!]_M(s) & = & \forall s'.\; R\;s\;s' \;\Rightarrow\; [\![\psi]\!]_M(s') \\
[\![\mathbf{EX}\psi]\!]_M(s) & = & \exists s'.\; R\;s\;s' \;\wedge\; [\![\psi]\!]_M(s') \\
[\![\mathbf{A}[\psi_1\;\mathbf{U}\;\psi_2]]\!]_M(s) & = & \forall\pi.\; \mathrm{Path}\;R\;s\;\pi \\
 & & \qquad \Rightarrow \exists i.\; [\![\psi_2]\!]_M(\pi(i)) \\
 & & \qquad\qquad\quad \wedge \\
 & & \qquad\qquad\quad \forall j.\; j{<}i \;\Rightarrow\; [\![\psi_1]\!]_M(\pi(j)) \\
[\![\mathbf{E}[\psi_1\;\mathbf{U}\;\psi_2]]\!]_M(s) & = & \exists\pi.\; \mathrm{Path}\;R\;s\;\pi \\
 & & \qquad \wedge\; \exists i.\; [\![\psi_2]\!]_M(\pi(i)) \\
 & & \qquad\qquad\quad \wedge \\
 & & \qquad\qquad\quad \forall j.\; j{<}i \;\Rightarrow\; [\![\psi_1]\!]_M(\pi(j))
\end{array}
$$

The defined operator **AF**

- Define $\mathbf{AF}\psi = \mathbf{A}[\top\ \mathbf{U}\ \psi]$
- $\mathbf{AF}\psi$ true at s iff ψ true somewhere on every R-path from s

$$
\begin{aligned}
[\![\mathbf{AF}\psi]\!]_M(s) &= [\![\mathbf{A}[\top\ \mathbf{U}\ \psi]]\!]_M(s) \\
&= \forall\pi.\ \mathrm{Path}\ R\ s\ \pi \\
&\qquad \Rightarrow \\
&\qquad \exists i.\ [\![\psi]\!]_M(\pi(i)) \ \wedge\ \forall j.\ j < i \ \Rightarrow\ [\![\top]\!]_M(\pi(j)) \\
&= \forall\pi.\ \mathrm{Path}\ R\ s\ \pi \\
&\qquad \Rightarrow \\
&\qquad \exists i.\ [\![\psi]\!]_M(\pi(i)) \ \wedge\ \forall j.\ j < i \ \Rightarrow\ \mathit{true} \\
&= \forall\pi.\ \mathrm{Path}\ R\ s\ \pi \ \Rightarrow\ \exists i.\ [\![\psi]\!]_M(\pi(i))
\end{aligned}
$$

The defined operator EF

- Define $\mathbf{EF}\psi = \mathbf{E}[\top\ \mathbf{U}\ \psi]$
- $\mathbf{EF}\psi$ true at s iff ψ true somewhere on some R-path from s

$$
\begin{aligned}
[\![\mathbf{EF}\psi]\!]_M(s) &= [\![\mathbf{E}[\top\ \mathbf{U}\ \psi]]\!]_M(s) \\
&= \exists\pi.\ \mathrm{Path}\ R\ s\ \pi \\
&\qquad \wedge \\
&\qquad \exists i.\ [\![\psi]\!]_M(\pi(i))\ \wedge\ \forall j.\ j < i\ \Rightarrow\ [\![\top]\!]_M(\pi(j)) \\
&= \exists\pi.\ \mathrm{Path}\ R\ s\ \pi \\
&\qquad \wedge \\
&\qquad \exists i.\ [\![\psi]\!]_M(\pi(i))\ \wedge\ \forall j.\ j < i\ \Rightarrow\ \mathit{true} \\
&= \exists\pi.\ \mathrm{Path}\ R\ s\ \pi\ \wedge\ \exists i.\ [\![\psi]\!]_M(\pi(i))
\end{aligned}
$$

- "can reach a state satisfying $p \in AP$" is $\mathbf{EF}\ p$

The defined operator **AG**

- Define $\mathbf{AG}\psi = \neg\mathbf{EF}(\neg\psi)$

- $\mathbf{AG}\psi$ true at s iff ψ true everywhere on every R-path from s

$$
\begin{aligned}
[\![\mathbf{AG}\psi]\!]_M(s) &= [\![\neg\mathbf{EF}(\neg\psi)]\!]_M(s) \\
&= \neg([\![\mathbf{EF}(\neg\psi)]\!]_M(s)) \\
&= \neg(\exists\pi.\ \mathrm{Path}\ R\ s\ \pi \wedge \exists i.\ [\![\neg\psi]\!]_M(\pi(i))) \\
&= \neg(\exists\pi.\ \mathrm{Path}\ R\ s\ \pi \wedge \exists i.\ \neg[\![\psi]\!]_M(\pi(i))) \\
&= \forall\pi.\ \neg(\mathrm{Path}\ R\ s\ \pi \wedge \exists i.\ \neg[\![\psi]\!]_M(\pi(i))) \\
&= \forall\pi.\ \neg\mathrm{Path}\ R\ s\ \pi \vee \neg(\exists i.\ \neg[\![\psi]\!]_M(\pi(i))) \\
&= \forall\pi.\ \neg\mathrm{Path}\ R\ s\ \pi \vee \forall i.\ \neg\neg[\![\psi]\!]_M(\pi(i)) \\
&= \forall\pi.\ \neg\mathrm{Path}\ R\ s\ \pi \vee \forall i.\ [\![\psi]\!]_M(\pi(i)) \\
&= \forall\pi.\ \mathrm{Path}\ R\ s\ \pi \Rightarrow \forall i.\ [\![\psi]\!]_M(\pi(i))
\end{aligned}
$$

- $\mathbf{AG}\psi$ means ψ true at all reachable states

- $[\![\mathbf{AG}(p)]\!]_M(s) \ \equiv\ \forall s'.\ R^*\ s\ s' \ \Rightarrow\ p \in L(s')$

- "can always reach a state satisfying $p \in AP$" is $\mathbf{AG}(\mathbf{EF}\ p)$

The defined operator **EG**

- Define $\mathbf{EG}\psi = \neg\mathbf{AF}(\neg\psi)$

- $\mathbf{EG}\psi$ true at s iff ψ true everywhere on some R-path from s

$$
\begin{aligned}
[\![\mathbf{EG}\psi]\!]_M(s) &= [\![\neg\mathbf{AF}(\neg\psi)]\!]_M(s) \\
&= \neg([\![\mathbf{AF}(\neg\psi)]\!]_M(s)) \\
&= \neg(\forall\pi.\ \mathrm{Path}\ R\ s\ \pi \Rightarrow \exists i.\ [\![\neg\psi]\!]_M(\pi(i))) \\
&= \neg(\forall\pi.\ \mathrm{Path}\ R\ s\ \pi \Rightarrow \exists i.\ \neg[\![\psi]\!]_M(\pi(i))) \\
&= \exists\pi.\ \neg(\mathrm{Path}\ R\ s\ \pi \Rightarrow \exists i.\ \neg[\![\psi]\!]_M(\pi(i))) \\
&= \exists\pi.\ \mathrm{Path}\ R\ s\ \pi \wedge \neg(\exists i.\ \neg[\![\psi]\!]_M(\pi(i))) \\
&= \exists\pi.\ \mathrm{Path}\ R\ s\ \pi \wedge \forall i.\ \neg\neg[\![\psi]\!]_M(\pi(i)) \\
&= \exists\pi.\ \mathrm{Path}\ R\ s\ \pi \wedge \forall i.\ [\![\psi]\!]_M(\pi(i))
\end{aligned}
$$

The defined operator $\mathbf{A}[\psi_1\ \mathbf{W}\ \psi_2]$

- $\mathbf{A}[\psi_1\ \mathbf{W}\ \psi_2]$ is a 'partial correctness' version of $\mathbf{A}[\psi_1\ \mathbf{U}\ \psi_2]$
- It is true at s if along all R-paths from s:
  - ψ_1 always holds on the path, or
  - ψ_2 holds sometime on the path, and until it does ψ_1 holds
- Define

$$
\begin{array}{l}
[\![\mathbf{A}[\psi_1\ \mathbf{W}\ \psi_2]]\!]_M(s) \\
\quad = [\![\neg\mathbf{E}[(\psi_1\wedge\neg\psi_2)\ \mathbf{U}\ (\neg\psi_1\wedge\neg\psi_2)]]\!]_M(s) \\
\quad = \neg[\![\mathbf{E}[(\psi_1\wedge\neg\psi_2)\ \mathbf{U}\ (\neg\psi_1\wedge\neg\psi_2)]]\!]_M(s) \\
\quad = \neg(\exists\pi.\ \mathrm{Path}\ R\ s\ \pi \\
\qquad\qquad\quad \wedge \\
\qquad\qquad\quad \exists i.\ [\![\neg\psi_1\wedge\neg\psi_2]\!]_M(\pi(i)) \\
\qquad\qquad\qquad\quad \wedge \\
\qquad\qquad\qquad\quad \forall j.\ j<i\ \Rightarrow\ [\![\psi_1\wedge\neg\psi_2]\!]_M(\pi(j)))
\end{array}
$$

- Exercise: understand the next two slides!

$\mathbf{A}[\psi_1\ \mathbf{W}\ \psi_2]$ continued (1)

- Continuing:

$$
\begin{array}{ll}
 & \neg(\exists \pi.\ \mathsf{Path}\ R\ s\ \pi \\
 & \qquad \wedge \\
 & \qquad \exists i.\ [\![\neg\psi_1 \wedge \neg\psi_2]\!]_M(\pi(i))\ \wedge\ \forall j.\ j<i\ \Rightarrow\ [\![\psi_1 \wedge \neg\psi_2]\!]_M(\pi(j))) \\
= & \forall \pi.\ \neg(\mathsf{Path}\ R\ s\ \pi \\
 & \qquad \wedge \\
 & \qquad \exists i.\ [\![\neg\psi_1 \wedge \neg\psi_2]\!]_M(\pi(i))\ \wedge\ \forall j.\ j<i\ \Rightarrow\ [\![\psi_1 \wedge \neg\psi_2]\!]_M(\pi(j))) \\
= & \forall \pi.\ \mathsf{Path}\ R\ s\ \pi \\
 & \qquad \Rightarrow \\
 & \qquad \neg(\exists i.\ [\![\neg\psi_1 \wedge \neg\psi_2]\!]_M(\pi(i))\ \wedge\ \forall j.\ j<i\ \Rightarrow\ [\![\psi_1 \wedge \neg\psi_2]\!]_M(\pi(j))) \\
= & \forall \pi.\ \mathsf{Path}\ R\ s\ \pi \\
 & \qquad \Rightarrow \\
 & \qquad \forall i.\ \neg[\![\neg\psi_1 \wedge \neg\psi_2]\!]_M(\pi(i)) \vee \neg(\forall j.\ j<i\ \Rightarrow\ [\![\psi_1 \wedge \neg\psi_2]\!]_M(\pi(j)))
\end{array}
$$

$\mathbf{A}[\psi_1\ \mathbf{W}\ \psi_2]$ continued (2)

- Continuing:

$$
\begin{array}{l}
= \forall \pi.\ \mathsf{Path}\ R\ s\ \pi \\
\qquad \Rightarrow \\
\qquad \forall i.\ \neg[\![\neg\psi_1 \wedge \neg\psi_2]\!]_M(\pi(i)) \vee \neg(\forall j.\ j{<}i\ \Rightarrow\ [\![\psi_1 \wedge \neg\psi_2]\!]_M(\pi(j)))
\end{array}
$$

$$
\begin{array}{l}
= \forall \pi.\ \mathsf{Path}\ R\ s\ \pi \\
\qquad \Rightarrow \\
\qquad \forall i.\ \neg(\forall j.\ j{<}i\ \Rightarrow\ [\![\psi_1 \wedge \neg\psi_2]\!]_M(\pi(j))) \vee \neg[\![\neg\psi_1 \wedge \neg\psi_2]\!]_M(\pi(i))
\end{array}
$$

$$
\begin{array}{l}
= \forall \pi.\ \mathsf{Path}\ R\ s\ \pi \\
\qquad \Rightarrow \\
\qquad \forall i.\ (\forall j.\ j{<}i\ \Rightarrow\ [\![\psi_1 \wedge \neg\psi_2]\!]_M(\pi(j)))\ \Rightarrow\ [\![\psi_1 \vee \psi_2]\!]_M(\pi(i))
\end{array}
$$

- Exercise: explain why this is $[\![\mathbf{A}[\psi_1\ \mathbf{W}\ \psi_2]]\!]_M(s)$?
    - this exercise illustrates the subtlety of writing CTL!

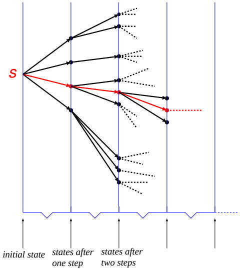
Sanity check: $\textbf{A}[\psi\ \textbf{W}\ \texttt{F}]\ =\ \textbf{AG}\ \psi$

- From last slide:
  $$\begin{array}{l}[\![\textbf{A}[\psi_1\ \textbf{W}\ \psi_2]]\!]_M(s)\\ \ =\ \forall\pi.\ \mathrm{Path}\ R\ s\ \pi\\ \qquad\qquad \Rightarrow\ \forall i.\ (\forall j.\ j{<}i \Rightarrow [\![\psi_1\wedge\neg\psi_2]\!]_M(\pi(j))) \Rightarrow [\![\psi_1\vee\psi_2]\!]_M(\pi(i))\end{array}$$
- Set ψ_1 to ψ and ψ_2 to $\texttt{F}$:
  $$\begin{array}{l}[\![\textbf{A}[\psi\ \textbf{W}\ \texttt{F}]]\!]_M(s)\\ \ =\ \forall\pi.\ \mathrm{Path}\ R\ s\ \pi\\ \qquad\qquad \Rightarrow\ \forall i.\ (\forall j.\ j{<}i\ \Rightarrow\ [\![\psi\wedge\neg\texttt{F}]\!]_M(\pi(j)))\ \Rightarrow\ [\![\psi\vee\texttt{F}]\!]_M(\pi(i))\end{array}$$
- Simplify:
  $$\begin{array}{l}[\![\textbf{A}[\psi\ \textbf{W}\ \texttt{F}]]\!]_M(s)\\ \ =\ \forall\pi.\ \mathrm{Path}\ R\ s\ \pi \Rightarrow \forall i.\ (\forall j.\ j{<}i \Rightarrow [\![\psi]\!]_M(\pi(j))) \Rightarrow [\![\psi]\!]_M(\pi(i))\end{array}$$
- By induction on i:
  $$[\![\textbf{A}[\psi\ \textbf{W}\ \texttt{F}]]\!]_M(s)\ =\ \forall\pi.\ \mathrm{Path}\ R\ s\ \pi\ \Rightarrow\ \forall i.\ [\![\psi]\!]_M(\pi(i))$$

- Exercises
  1. Describe the property: $\textbf{A}[\texttt{T}\ \textbf{W}\ \psi]$.
  2. Describe the property: $\neg\textbf{E}[\neg\psi_2\ \textbf{U}\ \neg(\psi_1\vee\psi_2)]$.
  3. Define $\textbf{E}[\psi_1\ \textbf{W}\ \psi_2] = \textbf{E}[\psi_1\ \textbf{U}\ \psi_2] \vee \textbf{EG}\psi_1$.
     Describe the property: $\textbf{E}[\psi_1\ \textbf{W}\ \psi_2]$?

Recall model behaviour computation tree

- Atomic properties are true or false of individual states
- General properties are true or false of whole behaviour
- Behaviour of (S, R) starting from $s \in S$ as a tree:



- A path is shown in red
- Properties may look at all paths, or just a single path
  - CTL: Computation Tree Logic (all paths from a state)
  - LTL: Linear Temporal Logic (a single path)

Summary of CTL operators (primitive + defined)

- CTL formulae:

p	(Atomic formula - $p \in AP$)
$\neg\psi$	(Negation)
$\psi_1 \wedge \psi_2$	(Conjunction)
$\psi_1 \vee \psi_2$	(Disjunction)
$\psi_1 \Rightarrow \psi_2$	(Implication)
$\mathbf{AX}\psi$	(All successors)
$\mathbf{EX}\psi$	(Some successors)
$\mathbf{AF}\psi$	(Somewhere – along all paths)
$\mathbf{EF}\psi$	(Somewhere – along some path)
$\mathbf{AG}\psi$	(Everywhere – along all paths)
$\mathbf{EG}\psi$	(Everywhere – along some path)
$\mathbf{A}[\psi_1\ \mathbf{U}\ \psi_2]$	(Until – along all paths)
$\mathbf{E}[\psi_1\ \mathbf{U}\ \psi_2]$	(Until – along some path)
$\mathbf{A}[\psi_1\ \mathbf{W}\ \psi_2]$	(Unless – along all paths)
$\mathbf{E}[\psi_1\ \mathbf{W}\ \psi_2]$	(Unless – along some path)

Example CTL formulae

- $\mathbf{EF}(\mathit{Started} \wedge \neg \mathit{Ready})$

  It is possible to get to a state where Started holds but Ready does not hold

- $\mathbf{AG}(\mathit{Req} \Rightarrow \mathbf{AF}\mathit{Ack})$

  If a request Req occurs, then it will eventually be acknowledged by Ack

- $\mathbf{AG}(\mathbf{AF}\mathit{DeviceEnabled})$

  DeviceEnabled is always true somewhere along every path starting anywhere: i.e. DeviceEnabled holds infinitely often along every path

- $\mathbf{AG}(\mathbf{EF}\mathit{Restart})$

  From any state it is possible to get to a state for which Restart holds

  Can't be expressed in LTL!

More CTL examples (1)

- $\mathbf{AG}(\mathit{Req} \Rightarrow \mathbf{A}[\mathit{Req}\ \mathbf{U}\ \mathit{Ack}])$

  If a request Req occurs, then it continues to hold, until it is eventually acknowledged

- $\mathbf{AG}(\mathit{Req} \Rightarrow \mathbf{AX}(\mathbf{A}[\neg\mathit{Req}\ \mathbf{U}\ \mathit{Ack}]))$

  Whenever Req is true either it must become false on the next cycle and remains false until Ack, or Ack must become true on the next cycle

  Exercise: is the **AX** necessary?

- $\mathbf{AG}(\mathit{Req} \Rightarrow (\neg\mathit{Ack} \Rightarrow \mathbf{AX}(\mathbf{A}[\mathit{Req}\ \mathbf{U}\ \mathit{Ack}])))$

  Whenever Req is true and Ack is false then Ack will eventually become true and until it does Req will remain true

  Exercise: is the **AX** necessary?

More CTL examples (2)

- $\mathbf{AG}(\mathit{Enabled} \Rightarrow \mathbf{AG}(\mathit{Start} \Rightarrow \mathbf{A}[\neg \mathit{Waiting}\ \mathbf{U}\ \mathit{Ack}]))$

  If Enabled is ever true then if Start is true in any subsequent state then Ack will eventually become true, and until it does Waiting will be false

- $\mathbf{AG}(\neg \mathit{Req}_1 \wedge \neg \mathit{Req}_2 \Rightarrow \mathbf{A}[\neg \mathit{Req}_1 \wedge \neg \mathit{Req}_2\ \mathbf{U}\ (\mathit{Start} \wedge \neg \mathit{Req}_2)])$

  Whenever Req_1 and Req_2 are false, they remain false until Start becomes true with Req_2 still false

- $\mathbf{AG}(\mathit{Req} \Rightarrow \mathbf{AX}(\mathit{Ack} \Rightarrow \mathbf{AF}\ \neg \mathit{Req}))$

  If Req is true and Ack becomes true one cycle later, then eventually Req will become false

Some abbreviations

- $\mathbf{AX}_i\ \psi \ \equiv \ \underbrace{\mathbf{AX}(\mathbf{AX}(\cdots(\mathbf{AX}\ \psi)\cdots))}_{i \text{ instances of } \mathbf{AX}}$

  ψ *is true on all paths* i *units of time later*

- $\mathbf{ABF}_{i..j}\ \psi \ \equiv \ \mathbf{AX}_i\,\underbrace{(\psi \vee \mathbf{AX}(\psi \vee \ \cdots \ \mathbf{AX}(\psi \vee \mathbf{AX}\ \psi)\cdots))}_{j-i \text{ instances of } \mathbf{AX}}$

  ψ *is true on all paths sometime between* i *units of time later and* j *units of time later*

- $\mathbf{AG}(Req \Rightarrow \mathbf{AX}(Ack_1 \wedge \mathbf{ABF}_{1..6}(Ack_2 \wedge \mathbf{A}[Wait\ \mathbf{U}\ Reply])))$

  One cycle after Req, Ack_1 *should become true, and then* Ack_2 *becomes true* 1 *to* 6 *cycles later and then eventually* $Reply$ *becomes true, but until it does* $Wait$ *holds from the time of* Ack_2

- More abbreviations in ‘Industry Standard’ language PSL

CTL model checking

- For LTL path formulae ϕ recall that $M \models \phi$ is defined by:

$$\boxed{M \models \phi \;\Leftrightarrow\; \forall \pi\ s.\ s \in S_0 \wedge \mathrm{Path}\ R\ s\ \pi \Rightarrow [\![\phi]\!]_M(\pi)}$$

- For CTL state formulae ψ the definition of $M \models \psi$ is:

$$\boxed{M \models \psi \;\Leftrightarrow\; \forall s.\ s \in S_0 \Rightarrow [\![\psi]\!]_M(s)}$$

- M common; LTL, CTL formulae and semantics $[\![\ ]\!]_M$ differ

- CTL model checking algorithm:
  - compute $\{s \mid [\![\psi]\!]_M(s) = \mathit{true}\}$ bottom up
  - check $S_0 \subseteq \{s \mid [\![\psi]\!]_M(s) = \mathit{true}\}$
  - symbolic model checking represents these sets as BDDs

CTL model checking: p, $\mathbf{AX}\psi$, $\mathbf{EX}\psi$

- For CTL formula ψ let $\{\!\!\{\psi\}\!\!\}_M = \{s \mid [\![\psi]\!]_M(s) = true\}$
- When unambiguous will write $\{\!\!\{\psi\}\!\!\}$ instead of $\{\!\!\{\psi\}\!\!\}_M$
- $\{\!\!\{p\}\!\!\} = \{s \mid p \in L(s)\}$
  - scan through set of states S marking states labelled with p
  - $\{\!\!\{p\}\!\!\}$ is set of marked states
- To compute $\{\!\!\{\mathbf{AX}\psi\}\!\!\}$
  - recursively compute $\{\!\!\{\psi\}\!\!\}$
  - marks those states all of whose successors are in $\{\!\!\{\psi\}\!\!\}$
  - $\{\!\!\{\mathbf{AX}\psi\}\!\!\}$ is the set of marked states
- To compute $\{\!\!\{\mathbf{EX}\psi\}\!\!\}$
  - recursively compute $\{\!\!\{\psi\}\!\!\}$
  - marks those states with at least one successor in $\{\!\!\{\psi\}\!\!\}$
  - $\{\!\!\{\mathbf{EX}\psi\}\!\!\}$ is the set of marked states

CTL model checking: $[\![\mathbf{E}[\psi_1\ \mathbf{U}\ \psi_2]]\!]$, $[\![\mathbf{A}[\psi_1\ \mathbf{U}\ \psi_2]]\!]$

- To compute $[\![\mathbf{E}[\psi_1\ \mathbf{U}\ \psi_2]]\!]$
  - recursively compute $[\![\psi_1]\!]$ and $[\![\psi_2]\!]$
  - mark all states in $[\![\psi_2]\!]$
  - mark all states in $[\![\psi_1]\!]$ with a successor state that is marked
  - repeat previous line until no change
  - $[\![\mathbf{E}[\psi_1\ \mathbf{U}\ \psi_2]]\!]$ is set of marked states

- More formally: $[\![\mathbf{E}[\psi_1\ \mathbf{U}\ \psi_2]]\!] = \bigcup_{n=0}^{\infty} [\![\mathbf{E}[\psi_1\ \mathbf{U}\ \psi_2]]\!]_n$ where:

$$
\begin{array}{lcl}
[\![\mathbf{E}[\psi_1\ \mathbf{U}\ \psi_2]]\!]_0 & = & [\![\psi_2]\!] \\
[\![\mathbf{E}[\psi_1\ \mathbf{U}\ \psi_2]]\!]_{n+1} & = & [\![\mathbf{E}[\psi_1\ \mathbf{U}\ \psi_2]]\!]_n \\
 & & \cup \\
 & & \{s \in [\![\psi_1]\!] \mid \exists s' \in [\![\mathbf{E}[\psi_1\ \mathbf{U}\ \psi_2]]\!]_n.\ R\ s\ s'\}
\end{array}
$$

- $[\![\mathbf{A}[\psi_1\ \mathbf{U}\ \psi_2]]\!]$ similar, but with a more complicated iteration
  - details omitted

Example: checking **EF** p

- $\mathbf{EF}p = \mathbf{E}[\top\ \mathbf{U}\ p]$
  - holds if ψ holds along some path

- Note $[\![\top]\!] = S$

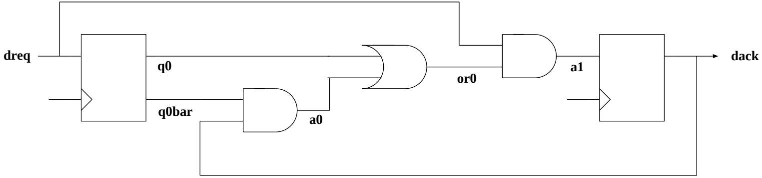
- Let $\mathcal{S}_n = [\![\mathbf{E}[\top\ \mathbf{U}\ p]]\!]_n$ then:

$$
\begin{array}{lcl}
\mathcal{S}_0 & = & [\![\mathbf{E}[\top\ \mathbf{U}\ p]]\!]_0 \\
 & = & [\![p]\!] \\
 & = & \{s \mid p \in L(s)\} \\
\mathcal{S}_{n+1} & = & \mathcal{S}_n \ \cup \ \{s \in [\![\top]\!] \mid \exists s' \in [\![\mathbf{E}[\top\ \mathbf{U}\ p]]\!]_n.\ R\ s\ s'\} \\
 & = & \mathcal{S}_n \ \cup \ \{s \mid \exists s' \in \mathcal{S}_n.\ R\ s\ s'\}
\end{array}
$$

  - mark all the states labelled with p
  - mark all with at least one marked successor
  - repeat until no change
  - $[\![\mathbf{EF}\ p]\!]$ is set of marked states

Example: RCV

- Recall the handshake circuit:



- State represented by a triple of Booleans $(dreq, q0, dack)$
- A model of RCV is M_{RCV} where:

  $M = (S_{\mathrm{RCV}}, S_{0\,\mathrm{RCV}}, R_{\mathrm{RCV}}, L_{\mathrm{RCV}})$

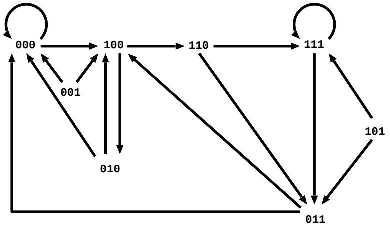
  and

  $$\begin{aligned} R_{\mathrm{RCV}}\ (dreq, q0, dack)\ (dreq', q0', dack') = \\ (q0' = dreq) \wedge (dack' = (dreq \wedge (q0 \vee dack))) \end{aligned}$$

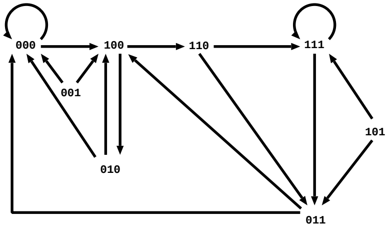
$\texttt{RCV}$ state transition diagram

- Possible states for $\texttt{RCV}$:
  $\{000, 001, 010, 011, 100, 101, 110, 111\}$
  where $b_2 b_1 b_0$ denotes state
  $\texttt{dreq} = b_2 \ \wedge\ \texttt{q0} = b_1 \ \wedge\ \texttt{dack} = b_0$

- Graph of the transition relation:



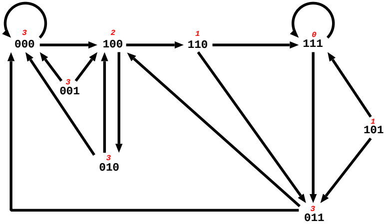
Computing $\{\mathbf{EF}\ \texttt{At111}\}$ where $\texttt{At111} \in L_{\texttt{RCV}}(s) \Leftrightarrow s = 111$



- Define:

$$
\begin{aligned}
\mathcal{S}_0 &= \{s \mid \texttt{At111} \in L_{\texttt{RCV}}(s)\} \\
&= \{s \mid s = 111\} \\
&= \{111\} \\[1em]
\mathcal{S}_{n+1} &= \mathcal{S}_n \ \cup\ \{s \mid \exists s' \in \mathcal{S}_n.\ \mathcal{R}(s, s')\} \\
&= \mathcal{S}_n \ \cup\ \{b_2 b_1 b_0 \mid \\
&\qquad\qquad \exists b_2' b_1' b_0' \in \mathcal{S}_n.\ (b_1' = b_2)\ \wedge\ (b_0' = b_2 \wedge (b_1 \vee b_0))\}
\end{aligned}
$$

Computing {**EF** `At111`} (continued)



- Compute:

$$
\begin{array}{lcl}
\mathcal{S}_0 & = & \{111\} \\
\mathcal{S}_1 & = & \{111\} \ \cup \ \{101, 110\} \\
 & = & \{111, 101, 110\} \\
\mathcal{S}_2 & = & \{111, 101, 110\} \ \cup \ \{100\} \\
 & = & \{111, 101, 110, 100\} \\
\mathcal{S}_3 & = & \{111, 101, 110, 100\} \ \cup \ \{000, 001, 010, 011\} \\
 & = & \{111, 101, 110, 100, 000, 001, 010, 011\} \\
\mathcal{S}_n & = & \mathcal{S}_3 \quad (n > 3)
\end{array}
$$

- $\{\textbf{EF}\ \texttt{At111}\} \ = \ \mathbb{B}^3 \ = \ S_{\texttt{RCV}}$
- $M_{\texttt{RCV}} \models \textbf{EF}\ \texttt{At111} \ \Leftrightarrow \ S_{0\,\texttt{RCV}} \subseteq S$

Symbolic model checking

- Represent sets of states with BDDs
- Represent Transition relation with a BDD
- If BDDs of $\{\psi\}$, $\{\psi_1\}$, $\{\psi_2\}$ are known, then:
  - BDDs of $\{\neg\psi\}$, $\{\psi_1 \wedge \psi_2\}$, $\{\psi_1 \vee \psi_2\}$, $\{\psi_1 \Rightarrow \psi_2\}$ computed using standard BDD algorithms
  - BDDs of $\{\mathbf{AX}\psi\}$, $\{\mathbf{EX}\psi\}$, $\{\mathbf{A}[\psi_1\ \mathbf{U}\ \psi_2]\}$, $\{\mathbf{E}[\psi_1\ \mathbf{U}\ \psi_2]]\}$ computed using straightforward algorithms (see textbooks)
- Model checking CTL generalises reachable states Iteration

History of Model checking

- CTL model checking due to Emerson, Clarke & Sifakis
- Symbolic model checking due to several people:
  - Clarke & McMillan (idea usually credited to McMillan's PhD)
  - Coudert, Berthet & Madre
  - Pixley
- SMV (McMillan) is a popular symbolic model checker:

`http://www.cs.cmu.edu/~modelcheck/smv.html`	(original)
`http://www.kenmcmil.com/smv.html`	(Cadence extension by McMillan)
`http://nusmv.irst.itc.it/`	(new implementation)

- Other temporal logics
  - CTL*: combines CTL and LTL
  - Engineer friendly industrial languages: PSL, SVA

Expressibility of CTL

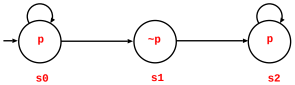
- Consider the property

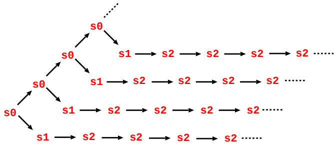
  "on every path there is a point after which p is always true on that path"

- Consider

  (($\star$) non-deterministically chooses `T` or `F`)

```
    0:  P:=1;
s0  1:  WHILE (*) DO SKIP;
s1  2:  P:=0;
s2  3:  P:=1;
    4:  WHILE T DO SKIP;
    5:
```





- Property true, but cannot be expressed in CTL
  - would need something like **AF**ψ
  - where ψ is something like "property p *true from now on*"
  - but in CTL ψ must start with a path quantifier **A** or **E**
  - cannot talk about current path, only about all or some paths
  - **AF(AG p)** is false (consider path $\mathtt{s0\,s0\,s0}\cdots$)

LTL can express things CTL can't

- Recall:
  $$\begin{array}{lcl} [\![\mathbf{F}\phi]\!]_M(\pi) & = & \exists i.\ [\![\phi]\!]_M(\pi{\downarrow}i) \\ [\![\mathbf{G}\phi]\!]_M(\pi) & = & \forall i.\ [\![\phi]\!]_M(\pi{\downarrow}i) \end{array}$$

- $\mathbf{FG}\phi$ is true if there is a point after which ϕ is always true
  $$\begin{array}{ll} [\![\mathbf{FG}\phi]\!]_M(\pi) & = [\![\mathbf{F}(\mathbf{G}(\phi))]\!]_M(\pi) \\ & = \exists m_1.\ [\![\mathbf{G}(\phi)]\!]_M(\pi{\downarrow}m_1) \\ & = \exists m_1.\ \forall m_2.\ [\![\phi]\!]_M((\pi{\downarrow}m_1){\downarrow}m_2) \\ & = \exists m_1.\ \forall m_2.\ [\![\phi]\!]_M(\pi{\downarrow}(m_1{+}m_2)) \end{array}$$

- LTL can express things that CTL can't express

- Note: it's tricky to prove CTL can't express $\mathbf{FG}\phi$

CTL can express things that LTL can't express

- **AG**(**EF** *p*) says:

  "from every state it is possible to get to a state for which p holds"

- Can't say this in LTL (easy proof given earlier)

- Consider disjunction:

  "along every path there is a state from which p will hold forever
  or
  from every state it is possible to get to a state for which p holds"

- Can't say this in either CTL or LTL!

- CTL* combines CTL and LTL and can express this property

CTL*

- Both state formulae (ψ) and path formulae (ϕ)
  - state formulae ψ are true of a state s like CTL
  - path formulae ϕ are true of a path π like LTL
- Defined mutually recursively

$$
\begin{array}{rcll}
\psi & ::= & p & \text{(Atomic formula)} \\
 & \mid & \neg\psi & \text{(Negation)} \\
 & \mid & \psi_1 \vee \psi_2 & \text{(Disjunction)} \\
 & \mid & \mathbf{A}\phi & \text{(All paths)} \\
 & \mid & \mathbf{E}\phi & \text{(Some paths)} \\
\phi & ::= & \psi & \text{(Every state formula is a path formula)} \\
 & \mid & \neg\phi & \text{(Negation)} \\
 & \mid & \phi_1 \vee \phi_2 & \text{(Disjunction)} \\
 & \mid & \mathbf{X}\phi & \text{(Successor)} \\
 & \mid & \mathbf{F}\phi & \text{(Sometimes)} \\
 & \mid & \mathbf{G}\phi & \text{(Always)} \\
 & \mid & [\phi_1\ \mathbf{U}\ \phi_2] & \text{(Until)}
\end{array}
$$

- CTL is CTL* with **X**, **F**, **G**, $[-\mathbf{U}-]$ preceded by **A** or **E**
- LTL consists of CTL* formulae of form $\mathbf{A}\phi$, where the only state formulae in ϕ are atomic

CTL* semantics

- Combines CTL state semantics with LTL path semantics:

$$
\begin{array}{lcl}
[\![p]\!]_M(s) & = & p \in L(s) \\
[\![\neg\psi]\!]_M(s) & = & \neg([\![\psi]\!]_M(s)) \\
[\![\psi_1 \vee \psi_2]\!]_M(s) & = & [\![\psi_1]\!]_M(s) \ \vee \ [\![\psi_2]\!]_M(s) \\
[\![\mathbf{A}\phi]\!]_M(s) & = & \forall\pi.\ \mathrm{Path}\ R\ s\ \pi \ \Rightarrow \ \phi(\pi) \\
[\![\mathbf{E}\phi]\!]_M(s) & = & \exists\pi.\ \mathrm{Path}\ R\ s\ \pi \ \wedge \ [\![\phi]\!]_M(\pi) \\
 & & \\
[\![\psi]\!]_M(\pi) & = & [\![\psi]\!]_M(\pi(0)) \\
[\![\neg\phi]\!]_M(\pi) & = & \neg([\![\phi]\!]_M(\pi)) \\
[\![\phi_1 \vee \phi_2]\!]_M(\pi) & = & [\![\phi_1]\!]_M(\pi) \ \vee \ [\![\phi_2]\!]_M(\pi) \\
[\![\mathbf{X}\phi]\!]_M(\pi) & = & [\![\phi]\!]_M(\pi{\downarrow}1) \\
[\![\mathbf{F}\phi]\!]_M(\pi) & = & \exists m.\ [\![\phi]\!]_M(\pi{\downarrow}m) \\
[\![\mathbf{G}\phi]\!]_M(\pi) & = & \forall m.\ [\![\phi]\!]_M(\pi{\downarrow}m) \\
[\![[\phi_1\ \mathbf{U}\ \phi_2]]\!]_M(\pi) & = & \exists i.\ [\![\phi_2]\!]_M(\pi{\downarrow}i) \wedge \forall j.\ j{<}i \ \Rightarrow \ [\![\phi_1]\!]_M(\pi{\downarrow}j)
\end{array}
$$

- Note $[\![\psi]\!]_M : S{\to}\mathbb{B}$ and $[\![\phi]\!]_M : (\mathbb{N}{\to}S){\to}\mathbb{B}$

LTL and CTL as CTL*

- As usual: $M = (S, S_0, R, L)$
- If ψ is a CTL* state formula: $M \models \psi \;\Leftrightarrow\; \forall s \in S_0.\ [\![\psi]\!]_M(s)$
- If ϕ is an LTL path formula then: $M \models_{\text{LTL}} \phi \;\Leftrightarrow\; M \models_{\text{CTL*}} \mathbf{A}\phi$
- If R is total ($\forall s.\ \exists s'.\ R\ s\ s'$) then (exercise):
  $\forall s\ s'.\ R\ s\ s' \;\Leftrightarrow\; \exists \pi.\ \text{Path}\ R\ s\ \pi \wedge (\pi(1) = s')$
- The meanings of CTL formulae are the same in CTL*

$$
\begin{array}{lll}
[\![\mathbf{A}(\mathbf{X}\psi)]\!]_M(s) & & \\
= \forall \pi.\ \text{Path}\ R\ s\ \pi \Rightarrow [\![\mathbf{X}\psi]\!]_M(\pi) & & \\
= \forall \pi.\ \text{Path}\ R\ s\ \pi \Rightarrow [\![\psi]\!]_M(\pi{\downarrow}1) & & (\psi \text{ as path formula}) \\
= \forall \pi.\ \text{Path}\ R\ s\ \pi \Rightarrow [\![\psi]\!]_M((\pi{\downarrow}1)(0)) & & (\psi \text{ as state formula}) \\
= \forall \pi.\ \text{Path}\ R\ s\ \pi \Rightarrow [\![\psi]\!]_M(\pi(1)) & &
\end{array}
$$

$$
\begin{array}{l}
[\![\mathbf{AX}\psi]\!]_M(s) \\
= \forall s'.\ R\ s\ s' \;\Rightarrow\; [\![\psi]\!]_M(s') \\
= \forall s'.\ (\exists \pi.\ \text{Path}\ R\ s\ \pi \wedge (\pi(1) = s')) \;\Rightarrow\; [\![\psi]\!]_M(s') \\
= \forall s'.\ \forall \pi.\ \text{Path}\ R\ s\ \pi \wedge (\pi(1) = s') \;\Rightarrow\; [\![\psi]\!]_M(s') \\
= \forall \pi.\ \text{Path}\ R\ s\ \pi \;\Rightarrow\; [\![\psi]\!]_M(\pi(1))
\end{array}
$$

Exercise: do similar proofs for other CTL formulae

Fairness

- May want to assume system or environment is 'fair'
- Example 1: fair arbiter
  the arbiter doesn't ignore one of its requests forever
  - not every request need be granted
  - want to exclude infinite number of requests and no grant
- Example 2: reliable channel
  no message continuously transmitted but never received
  - not every message need be received
  - want to exclude an infinite number of sends and no receive

Handling fairness in CTL and LTL

- Consider:
  p holds infinitely often along a path then so does q
- In LTL is expressible as $\mathbf{G}(\mathbf{F}\ p) \Rightarrow \mathbf{G}(\mathbf{F}\ q)$
- Can't say this in CTL
  - why not – what's wrong with $\mathbf{AG}(\mathbf{AF}\ p) \Rightarrow \mathbf{AG}(\mathbf{AF}\ q)$?
  - in CTL* expressible as $\mathbf{A}(\mathbf{G}(\mathbf{F}\ p) \Rightarrow \mathbf{G}(\mathbf{F}\ q))$
  - fair CTL model checking implemented in checking algorithm
  - fair LTL just a fairness assumption like $\mathbf{G}(\mathbf{F}\ p) \Rightarrow \cdots$
- Fairness is a tricky and subtle subject
  - many kinds of fairness:
    'weak fairness', 'strong fairness' etc
  - exist whole books on fairness

Propositional modal μ-calculus

- You may learn this in *Topics in Concurrency*
- μ-calculus is an even more powerful property language
  - has fixed-point operators
  - both maximal and minimal fixed points
  - model checking consists of calculating fixed points
  - many logics (e.g. CTL*) can be translated into μ-calculus
- Strictly stronger than CTL*
  - expressibility strictly increases as allowed nesting increases
  - need fixed point operators nested 2 deep for CTL*
- The μ-calculus is very non-intuitive to use!
  - intermediate code rather than a practical property language
  - nice meta-theory and algorithms, but terrible usability!

SEREs: Sequential Extended Regular Expressions

- SEREs are from the industrial PSL (more on PSL later)
- Syntax :

$$
\begin{array}{lcll}
r & ::= & p & \text{(Atomic formula } p \in AP\text{)} \\
 & | & !\,p & \text{(Negated atomic formula } p \in AP\text{)} \\
 & | & r_1 \mid r_2 & \text{(Disjunction)} \\
 & | & r_1 \;\&\&\; r_2 & \text{(Conjunction)} \\
 & | & r_1 \,;\, r_2 & \text{(Concatenation)} \\
 & | & r_1 \,:\, r_2 & \text{(Fusion)} \\
 & | & r[*] & \text{(Repeat)}
\end{array}
$$

- Semantics:
  (w ranges over finite lists of states s; $|w|$ is length of w; $w_1.w_2$ is concatenation; **head** w is head; $\langle\rangle$ is empty word)

$$
\begin{array}{ll}
[\![p]\!](w) & = p \in L(\textbf{head } w) \wedge |w| = 1 \\
[\![\,!\,p]\!](w) & = \neg(p \in L(\textbf{head } w)) \wedge |w| = 1 \\
[\![r_1 \mid r_2]\!](w) & = [\![r_1]\!](w) \ \vee \ [\![r_2]\!](w) \\
[\![r_1 \,\&\&\, r_2]\!](w) & = [\![r_1]\!](w) \ \wedge \ [\![r_2]\!](w) \\
[\![r_1 \,;\, r_2]\!](w) & = \exists w_1 \ w_2.\ w = w_1.w_2 \ \wedge \ [\![r_1]\!](w_1) \ \wedge \ [\![r_2]\!](w_2) \\
[\![r_1 : r_2]\!](w) & = \exists w_1 \ s \ w_2.\ w = w_1.s.w_2 \ \wedge \ [\![r_1]\!](w_1.s) \wedge [\![r_2]\!](s.w_2) \\
[\![r[*]]\!](w) & = w{=}\langle\rangle \vee \exists w_1 \cdots w_l.\ w{=}w_1.\cdots.w_l \wedge [\![r]\!](w_1) \wedge \cdots \wedge [\![r]\!](w_l)
\end{array}
$$

Example SERE

- Example

  A sequence in which `req` is asserted, followed four cycles later by an assertion of `grant`, followed by a cycle in which `abortin` is not asserted.

- Define p`[*3]` $= p;p;p$
- Then the example above can be represented by the SERE:

  `req;T[*3];grant;!abortin`

- In PSL this could be written as:

  `req;[*3];grant;!abortin`

  - where `[*3]` abbreviates `T[*3]`
  - more 'syntactic sugar' later
  - e.g. `true`, `false` for `T`, `F`

Assertion-Based Verification (ABV)

- It has been claimed that assertion based verification:

  "*is likely to be the next revolution in hardware design verification*"

- Basic idea:
  - document designs with formal properties
  - use simulation (dynamic) and model checking (static)
- Problem: too many languages
  - academic logics: LTL, CTL
  - tool-specific industrial versions:
    Intel, Cadence, Motorola, IBM, Synopsys
- What to do? Solution: a competition!
  - run by Accellera organisation
  - results standardised by IEEE
  - lots of politics

IBM's *Sugar* and Accellera's PSL

- *Sugar 1*: property language of IBM RuleBase checker
  - CTL plus *Sugar* *Extended Regular Expressions* (SEREs)
- Competition finalists: IBM's *Sugar 2* and Motorola's *CBV*
  - Intel/Synopsys ForSpec eliminated earlier
    (apparently industry politics involved)
- *Sugar 2* is based on LTL rather than CTL
  - has CTL constructs: "Optional Branching Extension" (OBE)
  - has clocking constructs for temporal abstraction
- Accellera purged "Sugar" from it property language
  - the word "Sugar" was too associated with IBM
  - language renamed to PSL
  - SEREs now *Sequential Extended Regular Expressions*
- Lobbying to make PSL more like ForSpec (align with SVA)

PSL Foundation Language (FL is LTL + SEREs)

- Syntax:

$$
\begin{array}{lcll}
f & ::= & p & \text{(Atomic formula - } p \in AP\text{)} \\
 & \mid & !\,f & \text{(Negation)} \\
 & \mid & f_1 \ \texttt{or}\ f_2 & \text{(Disjunction)} \\
 & \mid & \texttt{next}\ f & \text{(Successor)} \\
 & \mid & \{r\}(f) & \text{(Suffix implication: } r \text{ a SERE)} \\
 & \mid & \{r_1\}\ \texttt{|->}\ \{r_2\} & \text{(Suffix next implication: } r_1,\ r_2 \text{ SEREs)} \\
 & \mid & [f_1\ \texttt{until}\ f_2] & \text{(Until)}
\end{array}
$$

- Semantics (omits clocking, weak/strong distinction)

$$
\begin{array}{ll}
[\![p]\!]_M(\pi) & = p \in L(\pi(0)) \\
[\![!\,f]\!]_M(\pi) & = \neg([\![f]\!]_M(\pi)) \\
[\![f_1\ \texttt{or}\ f_2]\!]_M(\pi) & = [\![f_1]\!]_M(\pi) \ \vee\ [\![f_2]\!]_M(\pi) \\
[\![\texttt{next}\ f]\!]_M(\pi) & = [\![f]\!]_M(\pi{\downarrow}1) \\
[\![\{r\}(f)]\!]_M(\pi) & = \forall \pi'\ w.\ (\pi = w.\pi' \ \wedge\ [\![r]\!]_M(w)) \ \Rightarrow\ [\![f]\!]_M(\pi') \\
[\![\{r_1\}\texttt{|->}\{r_2\}]\!]_M(\pi) & = \forall \pi'\ w_1\ s.\ (\pi = w_1.s.\pi' \wedge [\![r_1]\!]_M(w_1.s)) \\
 & \qquad \Rightarrow \exists \pi''\ w_2.\ \pi' = w_2.\pi'' \wedge [\![r_2]\!]_M(s.w_2) \\
[\![[f_1\ \texttt{until}\ f_2]]\!]_M(\pi) & = \exists i.\ [\![f_2]\!]_M(\pi{\downarrow}i) \ \wedge\ \forall j.\ j<i \ \Rightarrow\ [\![f_1]\!]_M(\pi{\downarrow}j)
\end{array}
$$

- There is also an Optional Branching Extension (OBE)
  - completely standard CTL: **EX**, **E**[– – **U** – –], **EG** etc.

Combining SEREs with LTL formulae

- Formula $\{r\}f$ means LTL formula f true after SERE r
- Example

  After a sequence in which `req` *is asserted, followed four cycles later by an assertion of* `grant`*, followed by a cycle in which* `abortin` *is not asserted, we expect to see an assertion of* `ack` *some time in the future.*

- Can represent by

  ```
  always {req;[*3];grant;!abortin}(eventually ack)
  ```

- where `eventually` and `always` are defined by:

  ```
  eventually f   =  [true until f]
  always f       = !(eventually !f)
  ```

- N.B. Ignoring strong/weak distinction
  - strong/weak distinction important for dynamic checking
  - semantics when simulator halts before expected event
  - strictly should write `until!`, `eventually!`

SERE examples

- How can we modify

  `always reqin;ackout;!abortin |-> ackin;ackin`

  so that the two cycles of `ackin` start the cycle after `!abortin`

- Two ways of doing this

  ```
  always{reqin;ackout;!abortin}|->{true;ackin;ackin}
  always{reqin;ackout;!abortin}|=>{ackin;ackin}
  ```

- `|=>` is a defined operator

  `{r1}|=>{r2}  =  {r1}|->{true;r2}`

- Note: `true` and `T` are synonyms

Examples of defined notations: consecutive repetition

- Define

```
r[+]       = r;r[*]
              __
             |  false[*]  if i=0
r[*i]      = |
             |  r;...;r otherwise (i repetitions)
              __
r[*i..j] = r[*i] | r[*(i+1)] | ... | r[*j]
[+]        = true[+]
[*]        = true[*]
```

- Example

  Whenever we have a sequence of `req` *followed by* `ack`*, we should see a full transaction starting the following cycle. A full transaction starts with an assertion of the signal* `start_trans`*, followed by one to eight consecutive data transfers, followed by the assertion of signal* `end_trans`*. A data transfer is indicated by the assertion of signal* `data`

```
always{req;ack}|=>{start_trans;data[*1..8];end_trans}
```

Fixed number of non-consecutive repetitions

- Example

  Whenever we have a sequence of `req` followed by `ack`, we should see a full transaction starting the following cycle. A full transaction starts with an assertion of the signal `start_trans`, followed by eight not necessarily consecutive data transfers, followed by the assertion of signal `end_trans`. A data transfer is indicated by the assertion of signal `data`

- Can represent by

```
always
 {req;ack} |=>
 {start_trans;
  {{!data[*];data}[*8];!data[*]};
  end_trans}
```

- Define: `b[= i] = {!b[*];b}[*i];!b[*]`
- Then have a nicer representation

```
always{req;ack}|=>{start_trans;data[= 8];end_trans}
```

Variable number of non-consecutive repetitions

- Example

  Whenever we have a sequence of `req` *followed by* `ack`*, we should see a full transaction starting the following cycle. A full transaction starts with an assertion of the signal* `start_trans`*, followed by* $\boxed{\textit{one to eight}}$ *not necessarily consecutive data transfers, followed by the assertion of signal* `end_trans`*. A data transfer is indicated by the assertion of signal* `data`

- Define

```
b[= i..j] = {b[= i]} | {b[= (i+1)]} | ... | {b[= j]}
```

- Then

```
always {req;ack} |=>
       {start_trans;data[= 1..8];end_trans}
```

- These examples are meant to illustrate how PSL/Sugar is much more readable than raw CTL or LTL

Clocking

- Basic idea: `b@clk` samples `b` on rising edges of `clk`
- Can clock SEREs (`r@clk`) and formulae (`f@clk`)
- Can have several clocks
- Official semantics messy due to clocking
- Can 'translate away' clocks by pushing `@clk` inwards
  - rules given in PSL manual
  - roughly: `b@clk` $\leadsto$ `{!clk[*];clk & b}`

Model checking PSL (outline)

- SEREs checked by generating a finite automaton
  - recognise regular expressions
  - these automata are called “satellites”
- FL checked using standard LTL methods
- OBE checked by standard CTL methods
- Can also check formula for runs of a simulator
  - this is dynamic verification
  - semantics handles possibility of finite paths – messy!
- Commercial checkers only handle a subset of PSL

PSL layer structure

- Boolean layer has atomic predicates
- Temporal layer has LTL (FL) and CTL (OBE) properties
- Verification layer has commands for how to use properties
  - e.g. `assert`, `assume`

```
assert always (!en1 & en2))
   |      |          |
   |      |          |--- Boolean layer
   |      |
   |      |-------------- temporal layer
   |
   |--------------------- verification layer
```

- Modelling layer: HDL specification of e.g. inputs, checkers
  - e.g. augment `always(Req  ->  eventually! Ack)`
  - add counter to keep track of numbers of `Req` and `Ack`

PSL/Sugar summary

- Combines together LTL and CTL
- Regular expressions – SEREs
- LTL – Foundation Language formulae
- CTL – Optional Branching Extension
- Relatively simple set of primitives + definitional extension
- Boolean, temporal, verification, modelling layers
- Semantics for static and dynamic verification
  (needs strong/weak distinction)

Simulation semantics (a.k.a. event semantics)

- HDLs use *discrete event simulation*
  - changes to variables $\Rightarrow$ threads enabled
  - enabled threads executed non-deterministically
  - execution of threads $\Rightarrow$ more events
- Combinational thread:

  $\texttt{always @(}v_1\ \texttt{or}\ \cdots\ \texttt{or}\ v_n\texttt{)}\ v\texttt{:=}E$

  - enabled by any change to $v_1, \ldots, v_n$
- Positive edge triggered sequential threads:

  $\texttt{always @(posedge}\ clk\texttt{)}\ v\texttt{:=}E$

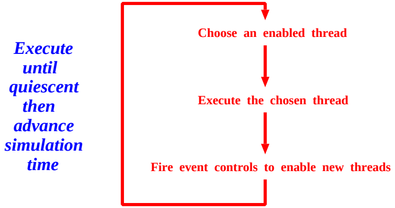
  - enabled by clk changing to `T`
- Negative edge triggered sequential threads:

  $\texttt{always @(negedge}\ clk\texttt{)}\ v\texttt{:=}E$

  - enabled by clk changing to `F`

Simulation

- Given
  - a set of threads
  - initial values for variables read or written by threads
  - a sequence of input values
    (inputs are variables not in LHS of assignments)
- *simulation algorithm* $\Rightarrow$ a sequence of states



- Simulation is non-deterministic

Combinational threads in series

$$in \longrightarrow \boxed{f} \xrightarrow{l_1} \boxed{g} \xrightarrow{l_2} \boxed{h} \longrightarrow out$$

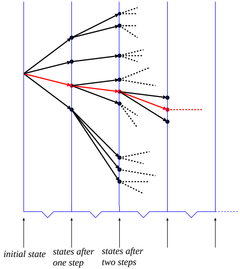
- HDL-like specification:

  `always @(`in`)` l_1 `:=` f(in) . . . . thread T1
  `always @(`l_1`)` l_2 `:=` g(l_1) . . . . thread T2
  `always @(`l_2`)` *out* `:=` h(l_2) . . . . thread T3

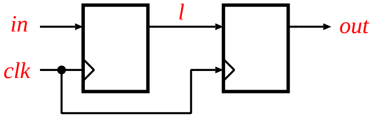
- Suppose *in* changes to `x` at simulation time *t*
  - T1 will become enabled and assign `f(x)` to l_1
  - if l_1's value changes then T2 will become enabled (still simulation time *t*)
  - T2 will assign `g(f(x))` to l_2
  - if l_2's value changes then T will become enabled (still simulation time *t*)
  - T3 will assign `h(g(f(x)))` to *out*
  - simulation quiesces (still simulation time *t*)
- Steps at same simulation time happen in "δ-*time*" (VHDL jargon)

Semantic gap

- Designers use HDLs and verify via simulation
  - event semantics
- Formal verifiers use logic and verify via proof
  - path semantics
- Problem: do path and simulation semantics agree?
- Would like:
  paths = sequences of quiescent simulation states



Sequential threads – event semantics

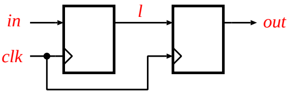


- Consider two Dtypes in series:

  `always @(posedge` *clk*`)` *l* `:=` *in*
  `always @(posedge` *clk*`)` *out* `:=` *l*

- If `posedge clk`:
  - both threads become enabled
  - race condition
- Right thread executed first:
  - *out* gets previous value of *l*
  - then left thread executed
  - so *l* gets value input at *in*
- Left thread executed first:
  - *l* gets input value at *in*
  - then right thread executed
  - so *out* gets input value at *in*

Sequential threads – path semantics



- Trace semantics:

```
in   aaaaaaaaaaabbbbbbccccccddddddddd......
clk  00000111110000011111000001111100......
l    eeeeeaaaaaaaaaabbbbbbbbbbddddddd......
out  fffffeeeeeeeeeeaaaaaaaaaabbbbbbb......
```

- Corresponds to right thread executed first
- How to ensure event and path semantics agree?
- **Method 1:** use non-blocking assignments:

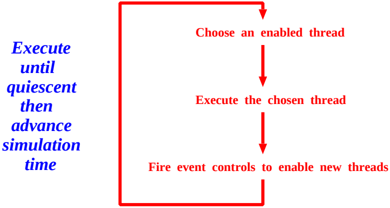
```
always @(posedge clk) l <= in;
always @(posedge clk) out <= l;
```

  - non-blocking assignments (<=) in Verilog
  - RHS of all non-blocking assignments first computed
  - assignments done at end of simulation cycle

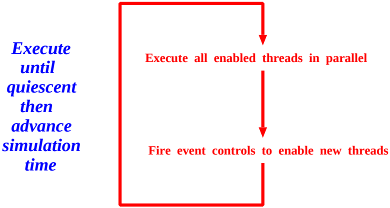
- **Method 2:** make simulation cycle VHDL-like

Verilog versus VHDL simulation cycles

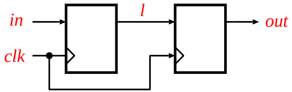
- Verilog-like simulation cycle:



- VHDL-like simulation cycle:



VHDL event semantics



- Recall HDL:
  `always @(posedge` *clk*`)` *l* `:=` *in*
  `always @(posedge` *clk*`)` *out* `:=` *l*
- If `posedge` *clk*:
  - both threads become enabled
- VHDL semantics:
  - both threads executed in parallel
  - *out* gets previous value of *l*
  - in parallel *l* gets value input at *in*
- Now no race
- Event semantics matches path semantics

Summary of dynamic versus static semantics

- Simulation (event) semantics different from path semantics
- No standard event semantics (Verilog versus VHDL)
- Verilog: need non-blocking assignments
- VHDL semantics closer path semantics
- Simulations are finite traces: better fit with LTL than CTL

Bisimulation equivalence: general idea

- M, M' bisimilar if they have 'corresponding executions'
  - to each step of M there is a corresponding step of M'
  - to each step of M' there is a corresponding step of M
- Bisimilar models satisfy same CTL* properties
- Bisimilar: same truth/falsity of model properties
- Simulation gives property-truth preserving abstraction (see later)

Bisimulation relations

- Let $R : S \to S \to \mathbb{B}$ and $R' : S' \to S' \to \mathbb{B}$ be transition relations
- B is a bisimulation relation between R and R' if:
  - $B : S \to S' \to \mathbb{B}$
  - $\forall s\ s'.\ B\ s\ s' \Rightarrow \forall s_1 \in S.\ R\ s\ s_1 \Rightarrow \exists s_1'.\ R'\ s'\ s_1' \wedge B\ s_1\ s_1'$
    (to each step of R there is a corresponding step of R')
  - $\forall s\ s'.\ B\ s\ s' \Rightarrow \forall s_1' \in S.\ R'\ s'\ s_1' \Rightarrow \exists s_1.\ R'\ s\ s_1 \wedge B\ s_1\ s_1'$
    (to each step of R' there is a corresponding step of R)

Bisimulation equivalence: definition and theorem

- Let $M = (S, S_0, R, L)$ and $M' = (S', S'_0, R', L')$
- $M \equiv M'$ if:
  - there is a bisimulation B between R and R'
  - $\forall s_0 \in S_0.\ \exists s'_0 \in S'_0.\ B\ s_0\ s'_0$
  - $\forall s'_0 \in S'_0.\ \exists s_0 \in S_0.\ B\ s_0\ s'_0$
  - there is a bijection $\theta : AP \to AP'$
  - $\forall s\ s'.\ B\ s\ s' \Rightarrow L(s) = L'(s')$
- Theorem: if $M \equiv M'$ then for any CTL* state formula ψ:
  $M \models \psi \ \Leftrightarrow\ M' \models \psi$
- See Q14 in the Exercises

Abstraction

- Abstraction creates a simplification of a model
  - separate states may get merged
  - an abstract path can represent several concrete paths
- $M \preceq \overline{M}$ means $\overline{M}$ is an abstraction of M
  - to each step of M there is a corresponding step of $\overline{M}$
  - atomic properties of M correspond to atomic properties of $\overline{M}$
- Special case is when $\overline{M}$ is a subset of M such that:
  - $\overline{M} = (\overline{S_0}, \overline{S}, \overline{R}, \overline{L})$ and $M = (S_0, S, R, L)$
    $\overline{S} \subseteq S$
    $\overline{S_0} = S_0$
    $\forall s\ s' \in \overline{S}.\ \overline{R}\ s\ s' \Leftrightarrow R\ s\ s'$
    $\forall s \in \overline{S}.\ \overline{L}\ s = L\ s$
  - $\overline{S}$ contain all reachable states of M
    $\forall s \in \overline{S}. \forall s' \in S.\ R\ s\ s' \Rightarrow s' \in \overline{S}$
- All paths of M from initial states are $\overline{M}$-paths
  - hence for all CTL formulas ψ: $\overline{M} \models \psi \ \Rightarrow M \models \psi$

Recall JM1

Thread 1	Thread 2
`0:  IF LOCK=0 THEN LOCK:=1;`	`0:  IF LOCK=0 THEN LOCK:=1;`
`1:  X:=1;`	`1:  X:=2;`
`2:  IF LOCK=1 THEN LOCK:=0;`	`2:  IF LOCK=1 THEN LOCK:=0;`
`3:`	`3:`

- Two program counters, state: $(pc_1, pc_2, lock, x)$

$$S_{\mathtt{JM1}} = [0..3] \times [0..3] \times \mathbb{Z} \times \mathbb{Z}$$

$$\begin{array}{lll|lll} R_{\mathtt{JM1}} & (0, pc_2, 0, x) & (1, pc_2, 1, x) & R_{\mathtt{JM1}} & (pc_1, 0, 0, x) & (pc_1, 1, 1, x) \\ R_{\mathtt{JM1}} & (1, pc_2, lock, x) & (2, pc_2, lock, 1) & R_{\mathtt{JM1}} & (pc_1, 1, lock, x) & (pc_1, 2, lock, 2) \\ R_{\mathtt{JM1}} & (2, pc_2, 1, x) & (3, pc_2, 0, x) & R_{\mathtt{JM1}} & (pc_1, 2, 1, x) & (pc_1, 3, 0, x) \end{array}$$

- Assume $\mathtt{NotAt11} \in L_{\mathtt{JM1}}(pc_1, pc_2, lock, x) \Leftrightarrow \neg((pc_1 = 1) \wedge (pc_2 = 1))$
- Model $M_{\mathtt{JM1}} = (S_{\mathtt{JM1}}, \{(0,0,0,0)\}, R_{\mathtt{JM1}}, L_{\mathtt{JM1}})$
- $S_{\mathtt{JM1}}$ not finite, but actually $lock \in \{0,1\}$, $x \in \{0,1,2\}$
- Clear by inspection that $M_{\mathtt{JM1}} \preceq \overline{M}_{\mathtt{JM1}}$ where:

$$\overline{M}_{\mathtt{JM1}} = (\overline{S}_{\mathtt{JM1}}, \{(0,0,0,0)\}, \overline{R}_{\mathtt{JM1}}, \overline{L}_{\mathtt{JM1}})$$

  - $\overline{S}_{\mathtt{JM1}} = [0..3] \times [0..3] \times [0..1] \times [0..3]$
  - $\overline{R}_{\mathtt{JM1}}$ is $R_{\mathtt{JM1}}$ restricted to arguments from $\overline{S}_{\mathtt{JM1}}$
  - $\mathtt{NotAt11} \in \overline{L}_{\mathtt{JM1}}(pc_1, pc_2, lock, x) \Leftrightarrow \neg((pc_1 = 1) \wedge (pc_2 = 1))$
  - $\overline{L}_{\mathtt{JM1}}$ is $L_{\mathtt{JM1}}$ restricted to arguments from $\overline{S}_{\mathtt{JM1}}$

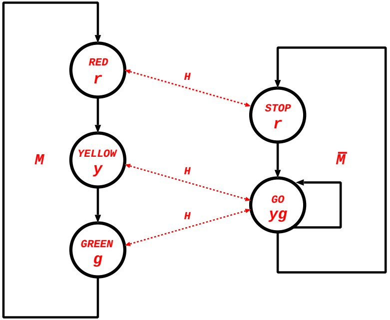
Simulation relations

- Let $R : S \to S \to \mathbb{B}$ and $\overline{R} : \overline{S} \to \overline{S} \to \mathbb{B}$ be transition relations
- H is a simulation relation between R and $\overline{R}$ if:
  - H is a relation between S and $\overline{S}$ – i.e. $H : S \to \overline{S} \to \mathbb{B}$
  - to each step of R there is a corresponding step of $\overline{R}$ – i.e.:
    $\forall s\ \overline{s}.\ H\ s\ \overline{s} \Rightarrow \forall s' \in S.\ R\ s\ s' \Rightarrow \exists \overline{s'} \in \overline{S}.\ \overline{R}\ \overline{s}\ \overline{s'} \wedge H\ s'\ \overline{s'}$
- Also need to consider abstraction of atomic properties
  - $H_{AP} : AP \to \overline{AP} \to \mathbb{B}$
  - details glossed over here

Simulation preorder: definition and theorem

- Let $M = (S, S_0, R, L)$ and $\overline{M} = (\overline{S}, \overline{S_0}, \overline{R}, \overline{L})$
- $M \preceq \overline{M}$ if:
  - there is a simulation H between R and $\overline{R}$
  - $\forall s_0 \in S_0.\ \exists \overline{s_0} \in \overline{S_0}.\ H\ s_0\ \overline{s_0}$
  - $\forall s\ \overline{s}.\ H\ s\ \overline{s} \Rightarrow L(s) = \overline{L}(\overline{s})$
- ACTL is the subset of CTL without **E**-properties
  - e.g. **AG AF**p – from anywhere can always reach a p-state
- Theorem: if $M \preceq \overline{M}$ then for any ACTL state formula ψ:
  $\overline{M} \models \psi \Rightarrow M \models \psi$
- If $\overline{M} \models \psi$ fails then cannot conclude $M \models \psi$ false

Example (Grumberg)



H a simulation

H *RED STOP* $\wedge$
H *YELLOW GO* $\wedge$
H *GREEN GO*

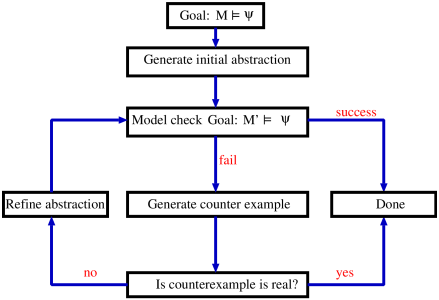
$H_{AP} : \{r, y, g\} \rightarrow \{r, yg\} \rightarrow \mathbb{B}$

$H_{AP}\ r\ r \wedge$
$H_{AP}\ y\ yg \wedge$
$H_{AP}\ g\ yg$

- $\overline{M} \models$ **AG AF** $\neg r$ hence $M \models$ **AG AF** $\neg r$
- but $\neg(\overline{M} \models$ **AG AF** $r)$ doesn't entail $\neg(M \models$ **AG AF** $r)$
  - $[\![$**AG AF** $r]\!]_{\overline{M}}(\mathit{STOP})$ is false
    (consider $\overline{M}$-path π' where $\pi' = \mathit{STOP.GO.GO.GO.}\cdots$)
  - $[\![$**AG AF** $r]\!]_{M}(\mathit{RED})$ is true
    (abstract path π' doesn't correspond to a real path in M)

CEGAR

- Counter Example Guided Abstraction Refinement



- Lots of details to fill out (several different solutions)
  - how to generate abstraction
  - how to check counterexamples
  - how to refine abstractions
- Microsoft SLAM driver verifier is a CEGAR system

Temporal Logic and Model Checking – Summary

- Various property languages: LTL, CTL, PSL (Prior, Pnueli)
- Models abstracted from hardware or software designs
- Model checking checks $M \models \psi$ (Clarke et al.)
- Symbolic model checking uses BDDs (McMillan)
- Avoid state explosion via simulation and abstraction
- CEGAR refines abstractions by analysing counterexamples
- Triumph of application of computer science theory
  - two Turing awards, McMillan gets 2010 CAV award
  - widespread applications in industry

THE END