4 Computation Theory (amp12)

(*a*) For the λ-calculus, define the notions of

(*i*) *β-conversion* ($=_\beta$) [2 marks]

(*ii*) *Church numeral* ($\underline{n}$) [2 marks]

(*b*) What does it mean for a total function $f : \mathbb{N}^n \to \mathbb{N}$ to be *λ-definable*? Explain why it is the case that not every $f : \mathbb{N}^n \to \mathbb{N}$ is λ-definable, carefully stating any standard results that you rely upon. [3 marks]

(*c*) Explain why the predecessor function $pred : \mathbb{N} \to \mathbb{N}$

$$pred(x) = \begin{cases} 0 & \text{if } x = 0 \\ x - 1 & \text{if } x > 0 \end{cases}$$

is λ-definable and give a λ-term that represents it. [4 marks]

(*d*) Show that the following functions are λ-definable. For each part you may assume solutions to the previous parts of the question.

(*i*) $if_0 : \mathbb{N}^3 \to \mathbb{N}$, where $if_0(x, y, z) = \begin{cases} y & \text{if } x = 0 \\ z & \text{if } x \neq 0 \end{cases}$ [3 marks]

(*ii*) $and : \mathbb{N}^2 \to \mathbb{N}$, where $and(x, y) = \begin{cases} 0 & \text{if } x = 0 \text{ and } y = 0 \\ 1 & \text{if } x \neq 0 \text{ or } y \neq 0 \end{cases}$ [1 mark]

(*iii*) $monus : \mathbb{N}^2 \to \mathbb{N}$, where $monus(x, y) = \begin{cases} x - y & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$ [3 marks]

(*iv*) $eq : \mathbb{N}^2 \to \mathbb{N}$, where $eq(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$ [2 marks]