6 Computation Theory (amp12)

(*a*) Explain why the Church-Rosser Theorem implies that any λ-term that is β-convertible ($=_\beta$) to a term in β-normal form is in fact β-reducible ($\twoheadrightarrow$) to one in β-normal form. [2 marks]

(*b*) Let *Bnf* denote the set of λ-terms that have a β-normal form. Give with justification an example of two closed λ-terms I and Ω with $\mathsf{I} \in \mathit{Bnf}$ and $\Omega \notin \mathit{Bnf}$. [3 marks]

(*c*) Suppose that $\#$ is a bijection between the set of all λ-terms and the set $\mathbb{N}$ of all natural numbers and that there are recursive functions $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $\nu : \mathbb{N} \to \mathbb{N}$ satisfying for all λ-terms M, N and numbers n that

$$\alpha(\#(M), \#(N)) = \#(M\,N) \qquad (1)$$
$$\nu(n) = \#(\underline{n}) \qquad (2)$$

where the λ-term $\underline{n}$ is the n^{th} Church numeral. Writing $\ulcorner M \urcorner$ for $\underline{\#(M)}$, show that there are closed λ-terms App and Num satisfying for all λ-terms M, N that

$$\mathsf{App}\,\ulcorner M \urcorner \ulcorner N \urcorner =_\beta \ulcorner M\,N \urcorner \qquad (3)$$
$$\mathsf{Num}\,\ulcorner N \urcorner =_\beta \ulcorner\ulcorner N \urcorner\urcorner \qquad (4)$$

(Any general properties of partial recursive functions with respect to λ-calculus you use should be carefully stated, but need not be proved.) [6 marks]

(*d*) Consider the following property of a closed λ-term F (where $\ulcorner M \urcorner$ is as in part (*c*)):

$$\text{for all } \lambda\text{-terms } M, \quad \begin{cases} F\,\ulcorner M \urcorner =_\beta \underline{0} & \text{if } M \in \mathit{Bnf} \\ F\,\ulcorner M \urcorner =_\beta \underline{1} & \text{if } M \notin \mathit{Bnf} \end{cases} \qquad (7)$$

Let $H = \lambda h.\mathsf{P}\,\Omega\,\mathsf{I}\,(F\,(\mathsf{App}\,h\,(\mathsf{Num}\,h)))$ where $\mathsf{P} = \lambda x\,y\,f.f\,(\lambda z.y)\,x$ and Ω and I are as in part (*b*). By considering whether or not $H\,\ulcorner H \urcorner$ is in *Bnf*, deduce that there can be no λ-term F satisfying (7). [6 marks]

(*e*) Deduce from part (*d*) that $\{\#(M) \mid M \in \mathit{Bnf}\}$ is an undecidable set of numbers. [3 marks]