9 Discrete Mathematics (mpf23)

(*a*) For sets A and B, recall that $A \Rightarrow B$ denotes the set of all functions from A to B and that $f : A \to B$ states that f is a function from A to B.

(*i*) Let R be a set.

For a set X define $\eta_X : X \to \big((X \Rightarrow R) \Rightarrow R\big)$ by

$$\eta_X(x)(f) \;=\; f(x)$$

and define $F : \Big(\big((X \Rightarrow R) \Rightarrow R\big) \Rightarrow R\Big) \to (X \Rightarrow R)$ by

$$F(\varphi)(x) \;=\; \varphi\big(\eta_X(x)\big)$$

Prove that F is surjective. [*Hint:* F is actually a retraction.] [6 marks]

(*ii*) Using the above, or otherwise, prove that for all sets X and R, if there is a surjection from X to $\Big(\big((X \Rightarrow R) \Rightarrow R\big) \Rightarrow R\Big)$ then R is a singleton. You may use standard results provided that you state them clearly. [4 marks]

(*b*) For sets Σ and A, let $a \in A$ and $f : \Sigma \times \Sigma^* \times A \to A$. Let R be the subset of $\Sigma^* \times A$ inductively defined by the axiom

$$\frac{}{(\,\varepsilon\,,\,a\,)}$$

and the rule

$$\frac{(\,w\,,\,x\,)}{\big(\,sw\,,\,f(s,w,x)\,\big)} \qquad (s \in \Sigma, w \in \Sigma^*, x \in A)$$

Prove that:

(*i*) R is total; that is, $\forall\, w \in \Sigma^*.\ \exists\, x \in A.\ (w,x) \in R.$ [4 marks]

(*ii*) R is functional; that is, $\forall\,(w,x) \in R.\ \forall\, y \in A.\ (w,y) \in R \;\Rightarrow\; y = x.$ [6 marks]