13 Types (nk480)

Recall the three judgements for classical propositional logic:

(*a*) $\Gamma; \Delta \vdash e : A \; \mathsf{true}$ – e is a proof of type A

(*b*) $\Gamma; \Delta \vdash k : A \; \mathsf{false}$ – k is a refutation of type A

(*c*) $\Gamma; \Delta \vdash \langle e \mid_A k \rangle \; \mathsf{contr}$ – $\langle e \mid_A k \rangle$ is a contradiction at type A

Here, Γ contains the true assumptions, and Δ are the false assumptions. In this question, we will extend classical propositional logic with support for the implication or function type operator $A \to B$.

(*a*) Give a proof term and inference rule for a proof of type $A \to B$. [4 marks]

(*b*) Give a proof term and inference rule for a refutation of type $A \to B$.

[*Hint*: how is implication encoded in classical logic?] [4 marks]

(*c*) Give a reduction rule for contradiction configurations of the form $\langle e \mid_{A \to B} k \rangle$. [4 marks]

(*d*) (*i*) State the preservation theorem for classical logic. [2 marks]

(*ii*) Give the proof of preservation for the case of the new rule defined above. You may assume that weakening, exchange and substitution all hold. [6 marks]