7 Foundations of Data Science (djw1005)

(*a*) Let $X_1, \ldots, X_n$ be independent binary random variables, $\mathbb{P}(X_i = 1) = \theta$, $\mathbb{P}(X_i = 0) = 1 - \theta$, for some unknown parameter θ. Using Uniform$[0, 1]$ as the prior distribution for θ, find the posterior distribution. [*Note:* For your answer, and in answer to parts (*b*) and (*d*), give either a named distribution with its parameters, or a normalised density function.] [3 marks]

I have collected a dataset of images, and employed an Amazon Mechanical Turk worker to label them. The labels are binary, nice or nasty. To assess how accurate the worker is, I first picked 30 validation images at random, found the true label myself, and compared the worker's label. The worker was correct on 25 and incorrect on 5.

(*b*) Let θ be the probability that the worker labels an image incorrectly. Using Beta$(0.1, 0.5)$ as the prior distribution for θ, find the posterior. [3 marks]

I next ask the worker to label a new test image, and they tell me the image is nice. Let $z \in \{\mathsf{nice}, \mathsf{nasty}\}$ be the true label, and let the prior distribution for z be $\Pr(\mathsf{nice}) = 0.1$, $\Pr(\mathsf{nasty}) = 0.9$.

(*c*) For both $z = \mathsf{nice}$ and $z = \mathsf{nasty}$, find

$$\mathbb{P}\big(\text{worker says } \mathsf{nice} \,|\, z, \theta\big).$$

Hence find the posterior distribution of (z, θ). Your answer may be left as an un-normalised density function. [5 marks]

(*d*) Find the posterior distribution of z. [5 marks]

My colleague has more grant money and she can employ 3 workers to rate each image. On a test set of 30 images, she found that they all agreed on 15 images, worker 1 was the odd one out on 8 of the images, worker 2 was the odd one out on 4, and worker 3 was the odd one out on 3.

(*e*) Let θ_i be the probability that worker i labels an image incorrectly. Find the posterior distribution of $(\theta_1, \theta_2, \theta_3)$. Your answer may be left as an un-normalised density function. [4 marks]

Hint. The Beta(α, β) *distribution has mean* $\alpha/(\alpha + \beta)$ *and density*

$$\Pr(x) = \binom{\alpha + \beta - 1}{\alpha - 1} x^{\alpha - 1} (1 - x)^{\beta - 1}, \quad x \in [0, 1].$$