COMPUTER SCIENCE TRIPOS Part II – 2017 – Paper 9

5 Denotational Semantics (MPF)

(*a*) (*i*) Give the grammar defining the PCF expressions that are values. [2 marks]

(*ii*) Prove or disprove that, for every PCF type τ, there is a closed PCF expression that is not a value of type τ. [2 marks]

(*b*) (*i*) Define the contextual-equivalence relation $M \cong_{\text{ctx}} N : \tau$ for pairs of closed PCF expressions M, N and a PCF type τ. [2 marks]

(*ii*) Prove or disprove that

$$(\mathbf{fn}\ n : nat.\, n) \cong_{\text{ctx}} \big(\mathbf{fn}\ n : nat.\, \mathbf{succ}(\mathbf{pred}(n))\big) : nat \to nat$$

[2 marks]

(*c*) For every pair of closed PCF expressions M, N of type nat, let $F_{M,N}$ be the closed PCF expression of type $(nat \to nat) \to (nat \to nat)$ given by

$$\begin{array}{l}\mathbf{fn}\ f : nat \to nat.\, \mathbf{fn}\ n : nat. \\ \quad \mathbf{if}\ \mathbf{zero}(n)\ \mathbf{then}\ M \\ \quad \mathbf{else\ if}\ \mathbf{zero}(\mathbf{pred}(n))\ \mathbf{then}\ N \\ \qquad \mathbf{else}\ \mathbf{succ}(f(\mathbf{pred}(n)))\end{array}$$

(*i*) Give an explicit description of $[\![\mathbf{fix}(F_{M,N})]\!] \in (\mathbb{N}_\perp \to \mathbb{N}_\perp)$ in terms of $[\![M]\!], [\![N]\!] \in \mathbb{N}_\perp$. Justify your answer. [8 marks]

(*ii*) Prove or disprove that there are closed PCF expressions M, N of type nat such that $\mathbf{fix}(F_{M,N}) \cong_{\text{ctx}} \big(\mathbf{fn}\ n : nat.\, \mathbf{pred}(n)\big) : nat \to nat$. You may use any standard results provided that you state them clearly. [4 marks]