UNIVERSITY OF CAMBRIDGE

COMPUTER SCIENCE TRIPOS Part IA

Tuesday 2 June 2015 1.30 to 4.30 pm

COMPUTER SCIENCE Paper 2

Answer ***one*** *question from each of Sections A, B and C, and* ***two*** *questions from Section D.*

Submit the answers in five ***separate*** *bundles, each with its own cover sheet. On each cover sheet, write the numbers of* ***all*** *attempted questions, and circle the number of the question attached.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

STATIONERY REQUIREMENTS
Script paper
Blue cover sheets
Tags
Rough work pad
Graph paper

SPECIAL REQUIREMENTS
Approved calculator permitted

SECTION A

1 Digital Electronics

(*a*) Write down simplified sum of products (SOP) and product of sums (POS) expressions for the following Boolean functions:

(*i*) $X = A \oplus B \oplus C$ [3 marks]

(*ii*) $Y = (A + \overline{B} + \overline{C}).(\overline{A} + \overline{D}).(A + C)$ [3 marks]

(*b*) Using a four variable Karnaugh map, fill it with 1s and 0s to find a function that illustrates each of the following situations. Write down the number of terms and the number of literals for each situation.

(*i*) The minimised SOP and POS forms have the same number of terms and literals. [3 marks]

(*ii*) The minimised POS form has fewer terms and literals than the minimised SOP form. [3 marks]

(*c*) For the following Boolean function,

$$F = \overline{A}.\overline{B}.\overline{C} + A.\overline{C}.\overline{D} + \overline{A}.C.D + B.C.\overline{D} + \overline{B}.C.D$$

show how it may be implemented using:

(*i*) one 16:1 multiplexor

(*ii*) one 8:1 multiplexor and one or more NOT gates

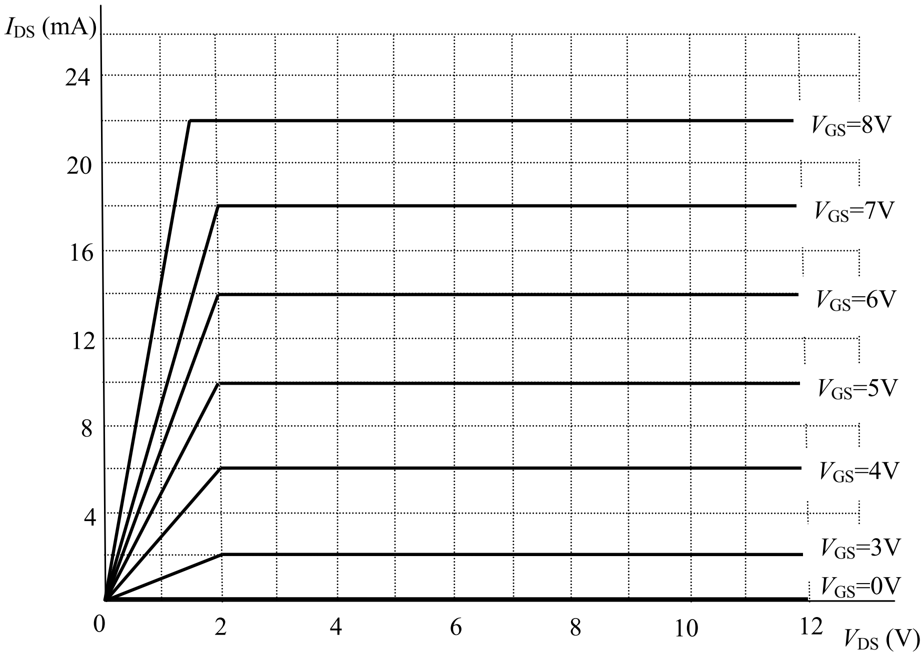
[8 marks]

2 Digital Electronics

(*a*) Draw a diagram showing the structure of an n-channel MOSFET and describe how the Drain to Source current can be controlled. [4 marks]

(*b*) Draw the circuit diagram of a NOT gate that comprises an n-channel MOSFET and a resistor R. [2 marks]

(*c*) For the NOT gate in part (*b*), plot the relationship between the input voltage, V_{in}, and the output voltage, V_{out}. The power supply voltage, $V_{\text{DD}} = 10$ V, $R = 500\ \Omega$, and the MOSFET has the characteristics given in the following figure. [6 marks]



(*d*) For the NOT gate in part (*c*), calculate the power dissipated by resistor R when $V_{\text{in}} = 8$ V. [3 marks]

(*e*) (*i*) Describe how the power dissipated by resistor R can be reduced. State any potential problems with your proposed solution. [3 marks]

(*ii*) Present a modified circuit for a NOT gate that eliminates the problem of static power dissipation in resistor R. [2 marks]

SECTION B

3 Operating Systems

(*a*) What is an *access control list*? [4 marks]

(*b*) What is a *capability* in the context of access control? [4 marks]

(*c*) How is file access control in Unix implemented, and what simplifications are made over one of the general mechanisms you have described above? [6 marks]

(*d*) What is the *principle of minimum privilege* and which approach, access control lists or capabilities, lends itself better to supporting a minimum privilege regime? [6 marks]

4 Operating Systems

(*a*) What is a *page fault*? [2 marks]

(*b*) What is a *segment fault*? What might be a sensible response to a segment fault on a stack segment? [2 marks]

(*c*) Describe the actions that follow after a page fault occurs. You should include those performed in hardware prior to a handler being started, and those that are taken within the handler. Your answer should include how segment faults might be detected and handled. [8 marks]

(*d*) What is *page thrashing* and how might it be avoided? [4 marks]

(*e*) Why is handling a page fault more complex than handling an interrupt or software trap? [4 marks]

SECTION C

5 Software and Interface Design

This question is concerned with the design of the embedded software in a device for switching audio and video streams in a lecture theatre. Correct operation of the overall system involves complex timing dependencies. For example, in this (fictional) system, the video projectors operate most reliably if the input signal is already connected when they power up, whereas the audio amplifier should only be powered up when there is no input signal present.

(*a*) Create outline sketches of two different UML diagrams that would be most useful in defining and refining the key design elements of this system as described above. [8 marks]

(*b*) Describe a formal specification approach that could be used to verify that individual components of the design, when combined, will exhibit the overall properties required. [6 marks]

(*c*) If it becomes necessary to replace a functional component after implementation of the system is complete, describe the approach that could be taken at different points in the resulting upgrade project to ensure that the modified system continues to exhibit the desired properties. [6 marks]

6 Software and Interface Design

This question is concerned with the user interface of a system for switching audio and video streams in a lecture theatre. This system should be usable by the following people:

- lecture theatre technical support staff who define standard configurations;
- regular lecturers who may personalise these configurations; and also
- visiting lecturers who might only use the system once.

(*a*) Create outline sketches of two different UML diagrams that would be most useful in defining and refining the key design elements of this system as described above. [8 marks]

(*b*) Describe an interaction design approach that can be used to determine whether the proposed user interface will exhibit the overall usability properties required. [6 marks]

(*c*) If it becomes necessary to replace a component of the user interface after implementation of the system is complete, describe the approach that could be taken at different points in the resulting upgrade project to ensure that the modified system continues to satisfy user needs. [6 marks]

SECTION D

7 Discrete Mathematics

(*a*) Let $\mathbb{N}_{\geq 2} \stackrel{\text{def}}{=} \{\, k \in \mathbb{N} \mid k \geq 2 \,\}$.

Without using the Fundamental Theorem of Arithmetic, prove that for all positive integers m and n,

$$\gcd(m,n) = 1 \iff \neg(\,\exists\, k \in \mathbb{N}_{\geq 2}.\ k \mid m \wedge k \mid n\,)$$

You may use any other standard results provided that you state them clearly. [6 marks]

(*b*) Recall that, for $i, j \in \mathbb{N}$,

$$\binom{i}{j} \stackrel{\text{def}}{=} \begin{cases} 0 & \text{, if } i < j \\ \frac{i!}{j!(i-j)!} & \text{, if } i \geq j \end{cases}$$

(*i*) Show that for all $m < l$ in $\mathbb{N}$,

$$\binom{l}{m+1} + \binom{l}{m} = \binom{l+1}{m+1}$$

[2 marks]

(*ii*) Prove that

$$\forall\, n \in \mathbb{N}.\ \forall\, m \in \mathbb{N}.\ 0 \leq m \leq n \implies \sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$$

[6 marks]

(*c*) Let U be a set and let $F : \mathbb{N} \times \mathbb{N} \to \mathcal{P}(U)$ be a function such that for all $i, i', j, j' \in \mathbb{N}$, if $i \leq i'$ and $j \leq j'$ then $F(i,j) \subseteq F(i',j')$ in $\mathcal{P}(U)$.

Prove that

$$\bigcup_{i \in \mathbb{N}} \left(\bigcup_{j \in \mathbb{N}} F(i,j) \right) = \bigcup_{k \in \mathbb{N}} F(k,k)$$

(Recall that $x \in \bigcup_{l \in L} X_l \iff \exists\, l \in L.\ x \in X_l$.) [6 marks]

8 Discrete Mathematics

(*a*) Prove that, for all natural numbers n,

$$n^{13} \equiv n \pmod{1365}$$

You may use any standard results provided that you state them clearly.

[5 marks]

(*b*) For n ranging over the natural numbers $\mathbb{N}$, let

Even(n) be the predicate $\exists\, k \in \mathbb{N}.\ n = 2 \cdot k$

and let

Odd(n) be the predicate $\exists\, l \in \mathbb{N}.\ n = 2 \cdot l + 1$

Prove that

$$\forall\, n \in \mathbb{N}.\ \mathrm{Even}(n) \vee \mathrm{Odd}(n)$$

by the Principle of Induction. [5 marks]

(*c*) Let $F : A \longrightarrow\!\!\!\!\!\!\!+\ B$ be a relation, from a set A to a set B.

(*i*) Define what it means for F to be a (total) function. [2 marks]

(*ii*) Prove that F is a function if, and only if, there exists a relation $G : B \longrightarrow\!\!\!\!\!\!\!+\ A$ such that $\mathrm{id}_A \subseteq G \circ F$ and $F \circ G \subseteq \mathrm{id}_B$. [8 marks]

9 Discrete Mathematics

(*a*) Without using the Fundamental Theorem of Arithmetic, prove that for all positive integers a, b, c,

$$\gcd(a, c) = 1 \implies \big(\gcd(a \cdot b, c) \mid b \wedge \gcd(a \cdot b, c) = \gcd(b, c)\big)$$

You may use any other standard results provided that you state them clearly. [6 marks]

(*b*) Prove that for all disjoint sets X and Y (that is, such that $X \cap Y = \emptyset$),

$$\mathcal{P}(X \cup Y) \cong \mathcal{P}(X) \times \mathcal{P}(Y)$$

You may use any standard results provided that you state them clearly. [6 marks]

(*c*) (*i*) For an alphabet Σ and a language $L \subseteq \Sigma^*$, define the language $F(L) \subseteq \Sigma^*$ as

$$F(L) \stackrel{\text{def}}{=} \{\, awa \in \Sigma^* \mid a \in \Sigma \wedge w \in L \,\}$$

Prove that for all $L_1, L_2 \subseteq \Sigma^*$, $F(L_1 \cup L_2) = F(L_1) \cup F(L_2)$. [2 marks]

(*ii*) Let $\text{Pal} \subseteq \Sigma^*$ be the language of palindromes (i.e. strings that read the same backwards as forwards) defined as $\text{Pal} \stackrel{\text{def}}{=} \{\, w \in \Sigma^* \mid \text{rev}(w) = w \,\}$ where rev is the unique function $\Sigma^* \to \Sigma^*$ such that $\text{rev}(\varepsilon) = \varepsilon$, $\text{rev}(a) = a$ for all $a \in \Sigma$, and $\text{rev}(w_1 w_2) = \text{rev}(w_2)\,\text{rev}(w_1)$ for all $w_1, w_2 \in \Sigma^*$.

Prove that for all $L \subseteq \Sigma^*$, $L \subseteq \text{Pal} \implies F(L) \subseteq \text{Pal}$. [2 marks]

(*iii*) For $k \in \mathbb{N}$, let $\text{Pal}_k \subseteq \text{Pal}$ be the language of palindromes of length k; that is, $\text{Pal}_k \stackrel{\text{def}}{=} \{w \in \text{Pal} \,\big|\, |w| = k\}$.

Recalling that $F^0(X) \stackrel{\text{def}}{=} X$ and, for $k \in \mathbb{N}$, $F^{k+1}(X) \stackrel{\text{def}}{=} F(F^k(X))$, prove that:

(A) for all $n \in \mathbb{N}$, $F^n(\text{Pal}_0) = \text{Pal}_{2n}$, [2 marks]

(B) for all $n \in \mathbb{N}$, $F^n(\text{Pal}_1) = \text{Pal}_{2n+1}$. [2 marks]

It follows that $\bigcup_{n\in\mathbb{N}} F^n(\text{Pal}_0 \cup \text{Pal}_1) = \text{Pal}$, though you are not required to prove this.

10 Discrete Mathematics

(*a*) Give a *deterministic* finite automaton (DFA) with input alphabet $\{a\}$ accepting the language $\{a^n \mid n \in U\}$, where $U = \{1, 2\} \cup \{n \geq 3 \mid n \equiv 4 \pmod 6 \vee n \equiv 7 \pmod 6\}$. [3 marks]

(*b*) What does it mean for a language over an alphabet Σ to be *regular*? [2 marks]

(*c*) A subset U of the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of natural numbers is called *ultimately periodic* if there exist numbers $N \geq 0$ and $p > 0$ such that for all $n \geq N$, $n \in U$ if and only if $n + p \in U$.

(*i*) Explain why every *finite* set of numbers is ultimately periodic according to the above definition. [2 marks]

(*ii*) Let L be a regular language over the alphabet $\{a\}$. By considering the shape of paths in the transition graph of any DFA with input alphabet $\{a\}$, or otherwise, show that $\{n \in \mathbb{N} \mid a^n \in L\}$ is an ultimately periodic set of numbers. [8 marks]

(*iii*) Conversely, show that if $U \subseteq \mathbb{N}$ is ultimately periodic, then $\{a^n \mid n \in U\}$ is a regular language. [5 marks]

END OF PAPER