4 Denotational Semantics (MPF)

Let τ be a PCF type.

(*a*) Consider the PCF terms

$$\begin{aligned} \mathbf{head} &: (nat \to \tau) \to \tau \\ \mathbf{tail} &: (nat \to \tau) \to nat \to \tau \\ \mathbf{repeat} &: (nat \to \tau) \to nat \to \tau \end{aligned}$$

given by the following definitions

$$\begin{aligned} \mathbf{head} &= \mathbf{fn}\, s : nat \to \tau.\, s(\mathbf{0}) \\ \mathbf{tail} &= \mathbf{fn}\, s : nat \to \tau.\, \mathbf{fn}\, n : nat.\, s(\mathbf{succ}\, n) \\ \mathbf{repeat} &= \mathbf{fix}\Big(\ \mathbf{fn}\, f : (nat \to \tau) \to nat \to \tau.\, \mathbf{fn}\, s : nat \to \tau.\, \mathbf{fn}\, n : nat. \\ &\qquad \mathbf{if}\ (\mathbf{zero}\, n)\ \mathbf{then}\ (\mathbf{head}\, s) \\ &\qquad \mathbf{else\ if}\ (\mathbf{zero}(\mathbf{pred}\, n))\mathbf{then}\ (\mathbf{head}\, s) \\ &\qquad \mathbf{else}\ f\, (\mathbf{tail}\, s)\, \big(\mathbf{pred}(\mathbf{pred}\, n)\big)\ \Big) \end{aligned}$$

Show that

$$[\![\, \mathbf{fn}\, s : nat \to \tau.\, \mathbf{tail}\big(\mathbf{tail}(\mathbf{repeat}\, s)\big)\,]\!] \;=\; [\![\, \mathbf{fn}\, s : nat \to \tau.\, \mathbf{repeat}(\mathbf{tail}\, s)\,]\!]$$

in the domain $\big((\mathbb{N}_\perp \to [\![\tau]\!]) \to (\mathbb{N}_\perp \to [\![\tau]\!])\big)$. [6 marks]

(*b*) Define a closed PCF term

$$\mathbf{shuffle} : (nat \to \tau) \to (nat \to \tau) \to nat \to \tau$$

such that

$$\begin{aligned} [\![\mathbf{head}]\!]\big([\![\mathbf{shuffle}]\!]\ s\ t\big) &= [\![\mathbf{head}\ s]\!] \\ [\![\mathbf{tail}]\!]\big([\![\mathbf{shuffle}]\!]\ s\ t\big) &= [\![\mathbf{shuffle}]\!]\ t\ ([\![\mathbf{tail}]\!]\ s) \end{aligned}$$

for all $s, t \in (\, \mathbb{N}_\perp \to [\![\tau]\!]\,)$. Briefly justify your answer. [5 marks]

(*c*) (*i*) Define the notion of least pre-fixed point $fix(f)$ in a domain D of a continuous function f in the function domain $(D \to D)$. [3 marks]

(*ii*) Prove that

$$[\![\mathbf{repeat}]\!] \sqsubseteq [\![\mathbf{fn}\, s : nat \to \tau.\, \mathbf{shuffle}\, s\, s]\!]$$

in the domain $\big((\mathbb{N}_\perp \to [\![\tau]\!]) \to (\mathbb{N}_\perp \to [\![\tau]\!])\big)$. [6 marks]