2 Complexity Theory (AD)

(*a*) State what it means for a graph $G = (V, E)$ to be 3-colourable. [2 marks]

(*b*) What is known about the complexity of deciding whether a given graph G is 3-colourable? [2 marks]

(*c*) Given a graph $G = (V, E)$ and a *partial* function $\chi : V \hookrightarrow \{1, 2, 3\}$, we define the graph G' by the following actions on G:

- for each pair $u, v \in V$ such that $\chi(u)$ and $\chi(v)$ are both defined and $\chi(u) \neq \chi(v)$, add an edge (u, v) to the graph; and
- for each pair $u, v \in V$ such that $\chi(u)$ and $\chi(v)$ are both defined and $\chi(u) = \chi(v)$, add new vertices w_1 and w_2 to the graph, along with the edges $(w_1, w_2), (u, w_1), (u, w_2), (v, w_1)$ and (v, w_2).

Prove that G' as constructed above is 3-colourable if, and only if, there is a valid 3-colouring of G that extends the partial function χ. [6 marks]

(*d*) Assume $\mathsf{P} = \mathsf{NP}$. Using this assumption and the construction in (*c*), describe a polynomial-time algorithm A which does the following:

> A takes as input a graph G. If G is not 3-colourable, A returns "no".
> If G is 3-colourable, A returns a valid 3-colouring of G.

[10 marks]