4 Denotational Semantics (AMP)

Given a closed PCF term F of type $nat \to nat$ and a function $f : \mathbb{N} \to \mathbb{N}$, say that F *represents* f if $F(\mathbf{succ}^n(\mathbf{0})) \Downarrow_{nat} \mathbf{succ}^{f(n)}(\mathbf{0})$ holds for all $n \in \mathbb{N}$.

(*a*) What is the *soundness* property of the denotational semantics of PCF? Use it to show that if f is not a constant function (that is, $f(m) \neq f(n)$ for some $m \neq n$), then the denotation $[\![F]\!] : \mathbb{N}_\perp \to \mathbb{N}_\perp$ of any F that represents f is the strict function that equals f when restricted to $\mathbb{N}$. [4 marks]

(*b*) If f is a constant function ($f(n) = c$ for all n, say), give, with justification, two PCF terms that represent it and that are not contextually equivalent. [5 marks]

(*c*) Consider the PCF term

$$G \stackrel{\text{def}}{=} \mathbf{fix}(\mathbf{fn}\, x : nat \to nat\, .\, \mathbf{fn}\, y : nat\, .\, \mathbf{ifzero}(F\, y)\, \mathbf{then}\, y\, \mathbf{else}\, x(\mathbf{succ}(y)))$$

where F represents a function $f : \mathbb{N} \to \mathbb{N}$ with the property that $f(n) = 0$ holds for infinitely many $n \in \mathbb{N}$. Let $\Phi : (\mathbb{N}_\perp \to \mathbb{N}_\perp) \to (\mathbb{N}_\perp \to \mathbb{N}_\perp)$ be the continuous function whose least fixed point is $[\![G]\!]$. Show by induction on k that for all $k, n \in \mathbb{N}$

$$\Phi^k(\perp)(n) = \begin{cases} \text{least } m \text{ such that } n \leq m < n + k \text{ and } f(m) = 0 \\ \perp \quad \text{if no such } m \text{ exists.} \end{cases}$$

[4 marks]

(*d*) State the *adequacy* property of the denotational semantics of PCF and *Tarski's Fixed Point Theorem* for continuous functions on a domain. Use them to deduce that the term G in part (c) represents the function $\mu_f : \mathbb{N} \to \mathbb{N}$ that maps each $n \in \mathbb{N}$ to the least $m \geq n$ such that $f(m) = 0$. [7 marks]