14 Types (AMP)

(*a*) For each type variable α, the *option type* O_α is defined to be the PLC type

$$O_\alpha \stackrel{\text{def}}{=} \forall \beta\, (\beta \to (\alpha \to \beta) \to \beta).$$

Prove that there are closed PLC expressions *None*, *Some* and *Case* with the following typing and beta-conversion properties.

(*i*) $\vdash \mathit{None} : \forall \alpha\, (O_\alpha)$

(*ii*) $\vdash \mathit{Some} : \forall \alpha\, (\alpha \to O_\alpha)$

(*iii*) $\vdash \mathit{Case} : \forall \alpha, \beta\, (\beta \to (\alpha \to \beta) \to O_\alpha \to \beta)$

(*iv*) $\mathit{Case}\, \alpha\, \beta\, y\, f\, (\mathit{None}\, \alpha) =_\beta y$

(*v*) $\mathit{Case}\, \alpha\, \beta\, y\, f\, (\mathit{Some}\, \alpha\, x) =_\beta f\, x.$

[10 marks]

(*b*) Use *Case* and *None* to define a closed PLC expression *Lift* of type $\forall \alpha_1, \alpha_2\, ((\alpha_1 \to O_{\alpha_2}) \to O_{\alpha_1} \to O_{\alpha_2})$ with the property that for all closed types τ and all closed expressions M of type $O_\alpha[\tau/\alpha]$, $\mathit{Lift}\, \tau\, \tau (\mathit{Some}\, \tau)\, M =_\beta M$. You may assume that any closed beta-normal form of type $O_\alpha[\tau/\alpha]$ is beta-convertible either to $\mathit{None}\, \tau$, or to $\mathit{Some}\, \tau\, N$ where N is a beta-normal form of type τ. Any standard results about PLC that you use should be carefully stated. [10 marks]