6 Denotational Semantics (AMP)

(*a*) If D and D' are domains, explain what is the *function domain* $D \to D'$; give its partial order and least element, and explain how least upper bounds of chains are calculated in it. [4 marks]

(*b*) An element d of a domain D is said to be *isolated* if for all countable chains $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \ldots$ in D with $d \sqsubseteq \bigsqcup_{n \geq 0} x_n$, there exists $i \geq 0$ with $d \sqsubseteq x_i$. We write $K(D)$ for the subset of isolated elements.

Given domains D and D' and elements $d \in D$ and $d' \in D'$, let $[d, d'] : D \to D'$ be the function mapping each $x \in D$ to d' if $d \sqsubseteq x$ and to $\bot$ otherwise.

(*i*) Prove that $[d, d']$ is monotone. [2 marks]

(*ii*) Prove that if $f : D \to D'$ is monotone, then $[d, d'] \sqsubseteq f$ if and only if $d' \sqsubseteq f(d)$. [2 marks]

(*iii*) Prove that if $d \in K(D)$, then $[d, d']$ is an element of the function domain $D \to D'$. [3 marks]

(*iv*) Prove that if both $d \in K(D)$ and $d' \in K(D')$, then $[d, d']$ is an isolated element of the function domain $D \to D'$. [3 marks]

(*v*) Now suppose that every element of D is the least upper bound of some countable chain of isolated elements and the same is true for D'. Show that each element f of the function domain $D \to D'$ is the least upper bound of the subset $F \stackrel{\text{def}}{=} \{[d, d'] \mid d \in K(D) \;\&\; d' \in K(D') \;\&\; d' \sqsubseteq f(d)\}$. [6 marks]