9 Semantics of Programming Languages (SS)

This question is about a very simple programming language with support for exceptions. The syntax and semantics is given to you.

The language has the following syntax.

Exceptions:	$\gamma, \gamma_1, \gamma_2, \ldots$
Expressions:	$e ::= \mathsf{true} \mid \mathsf{false} \mid \mathsf{if}\ e_1\ \mathsf{then}\ e_2\ \mathsf{else}\ e_3$
	$\mid e_1\ \mathsf{handle}\ \gamma \Rightarrow e_2 \mid \mathsf{raise}\,\gamma$
Types:	There is only one type: bool

A *typing context* is a set Γ of exceptions. The typing rules are as follows:

$$\frac{}{\Gamma \vdash \mathsf{true} : \mathsf{bool}} \qquad \frac{}{\Gamma \vdash \mathsf{false} : \mathsf{bool}} \qquad \frac{\Gamma \vdash e_1 : \mathsf{bool} \qquad \Gamma \vdash e_2 : \mathsf{bool} \quad \Gamma \vdash e_3 : \mathsf{bool}}{\Gamma \vdash \mathsf{if}\ e_1\ \mathsf{then}\ e_2\ \mathsf{else}\ e_3 : \mathsf{bool}}$$

$$\frac{\Gamma \cup \{\gamma\} \vdash e_1 : \mathsf{bool} \qquad \Gamma \vdash e_2 : \mathsf{bool}}{\Gamma \vdash e_1\ \mathsf{handle}\ \gamma \Rightarrow e_2 : \mathsf{bool}}\ (\gamma \notin \Gamma) \qquad \frac{}{\Gamma \vdash \mathsf{raise}\,\gamma : \mathsf{bool}}\ (\gamma \in \Gamma)$$

The reduction relation $\longrightarrow$ is defined as follows:

$$\frac{}{\mathsf{if\ true\ then}\ e_2\ \mathsf{else}\ e_3 \longrightarrow e_2} \qquad \frac{}{\mathsf{if\ false\ then}\ e_2\ \mathsf{else}\ e_3 \longrightarrow e_3}$$

$$\frac{e_1 \longrightarrow e_1'}{\mathsf{if}\ e_1\ \mathsf{then}\ e_2\ \mathsf{else}\ e_3 \longrightarrow \mathsf{if}\ e_1'\ \mathsf{then}\ e_2\ \mathsf{else}\ e_3}$$

$$\frac{}{(\mathsf{true\ handle}\ \gamma \Rightarrow e_2) \longrightarrow \mathsf{true}} \qquad \frac{}{(\mathsf{false\ handle}\ \gamma \Rightarrow e_2) \longrightarrow \mathsf{false}}$$

$$\frac{}{((\mathsf{raise}\,\gamma)\ \mathsf{handle}\ \gamma \Rightarrow e_2) \longrightarrow e_2} \qquad \frac{e_1 \longrightarrow e_1'}{(e_1\ \mathsf{handle}\ \gamma \Rightarrow e_2) \longrightarrow (e_1'\ \mathsf{handle}\ \gamma \Rightarrow e_2)}$$

$$\frac{}{\mathsf{if}\ (\mathsf{raise}\,\gamma)\ \mathsf{then}\ e_2\ \mathsf{else}\ e_3 \longrightarrow \mathsf{raise}\,\gamma} \qquad \frac{}{((\mathsf{raise}\,\gamma)\ \mathsf{handle}\ \gamma' \Rightarrow e_2) \longrightarrow \mathsf{raise}\,\gamma}\ (\gamma \neq \gamma')$$

(*a*) Consider the program $e_0 \stackrel{\text{def}}{=} \mathsf{if}\ ((\mathsf{raise}\,\gamma)\ \mathsf{handle}\ \gamma \Rightarrow \mathsf{true})\ \mathsf{then\ false\ else\ true}$.

(*i*) Give a derivation for $\vdash e_0 : \mathsf{bool}$. [4 marks]

(*ii*) Give a derivation for all of the transition steps for e_0. [3 marks]

(*b*) Prove the following theorem about this language:

If $\Gamma \vdash e : \mathsf{bool}$ then one of the following four conditions holds: (1) $e = \mathsf{true}$; (2) $e = \mathsf{false}$; (3) there is $\gamma \in \Gamma$ such that $e = (\mathsf{raise}\,\gamma)$; (4) there is e' such that $e \longrightarrow e'$. [13 marks]