4 Computation Theory (AMP)

(*a*) Define what it means for a set of numbers $S \subseteq \mathbb{N}$ to be register machine *decidable*. Why are there only countably many such sets? Deduce the existence of a set of numbers that is not register machine decidable. (Any standard results that you use should be clearly stated.) [4 marks]

(*b*) A set of numbers $S \subseteq \mathbb{N}$ is said to be *computably enumerable* if either it is empty or equal to $\{f(x) \mid x \in \mathbb{N}\}$ for some total function $f : \mathbb{N} \to \mathbb{N}$ that is register machine computable.

(*i*) Show that if S is register machine decidable, then it is computably enumerable. [*Hint:* consider separately the cases when S is, or is not empty.] [4 marks]

(*ii*) Show that if both S and its complement $\{x \in \mathbb{N} \mid x \notin S\}$ are computably enumerable, then S is register machine decidable. [*Hint:* consider a register machine that interleaves the enumeration of S and its complement.] [6 marks]

(*c*) Let $\varphi_e : \mathbb{N} \rightharpoonup \mathbb{N}$ denote the partial function computed by the register machine with code $e \in \mathbb{N}$ and consider the set $T = \{e \in \mathbb{N} \mid \varphi_e \text{ is a total function}\}$.

(*i*) Suppose that $f : \mathbb{N} \to \mathbb{N}$ is a register machine computable total function such that $f(x) \in T$ for all $x \in \mathbb{N}$. Define $\hat{f}(x)$ to be $\varphi_{f(x)}(x) + 1$. Show that $\hat{f} = \varphi_e$ for some $e \in T$. [3 marks]

(*ii*) Deduce that T is not computably enumerable. [3 marks]