2011 Paper 2 Question 7

Probability

(a) State the *probability mass function* for a Poisson random variable with parameter $\lambda > 0$. [2 marks]

(b) Define the *probability generating function*, $G_X(z)$, of a random variable X taking values in $\{0, 1, 2, \ldots\}$ and derive an expression for $G_X(z)$ in the case where $X \sim \text{Pois}(\lambda)$ with $\lambda > 0$. [4 marks]

(c) Show the following result

$$G_X^{(r)}(1) = E(X(X-1)\cdots(X-r+1))$$

where r is a positive integer and $G_X^{(r)}(1)$ denotes the rth derivative of $G_X(z)$ with respect to z evaluated at $z = 1$. [4 marks]

(d) Using the result in part (c) derive the mean and variance of a Poisson random variable with parameter $\lambda > 0$. [4 marks]

(e) Show the result that if X and Y are two independent random variables with probability generating functions $G_X(z)$ and $G_Y(z)$, respectively, then

$$G_{X+Y}(z) = G_X(z)G_Y(z)$$

where $G_{X+Y}(z)$ is the probability generating function of $X + Y$. [2 marks]

(f) Show that if $\lambda_1, \lambda_2 > 0$ and $X \sim \text{Pois}(\lambda_1)$ and $Y \sim \text{Pois}(\lambda_2)$ are independent random variables then $X+Y \sim \text{Pois}(\lambda_1+\lambda_2)$. What are the mean and variance of $X + Y$? [4 marks]