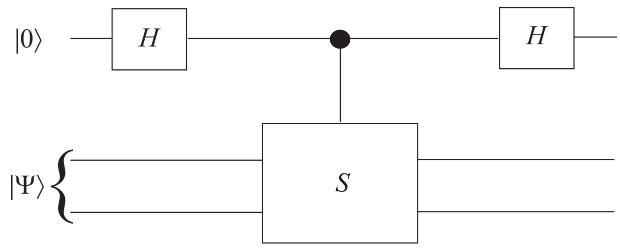
2009 Paper 9 Question 11

Quantum Computing

(*a*) The SWAP-gate is a two-qubit quantum gate implementing the unitary operation $S: |\psi\rangle|\phi\rangle \mapsto |\phi\rangle|\psi\rangle$, for all arbitrary states $|\psi\rangle$ and $|\phi\rangle$ in $\mathbb{C}^2$.

(*i*) Write the matrix for S, in the computational basis. [3 marks]

(*ii*) Consider the quantum circuit and inputs depicted below. Write the final (output) state in terms of $(|\Psi\rangle + S|\Psi\rangle)$ and $(|\Psi\rangle - S|\Psi\rangle)$. [2 marks]



(*iii*) Let $p_0(|\Psi\rangle)$ denote the probability of getting outcome "0" if, after the final Hadamard gate in the above circuit with input $|0\rangle|\Psi\rangle$, you measure the top qubit with respect to the computational basis. Compute $p_0(|B\rangle)$ for each Bell state $|B\rangle$. [2 marks]

(*iv*) Suppose you are given two qubits, whose states are denoted $|\psi\rangle$ and $|\phi\rangle$, and you are promised that, with equal probability, $|\psi\rangle$ and $|\phi\rangle$ are either *identical* or *orthogonal*. (1) Describe how to use the above quantum circuit and measurement to help you decide which *case* – identical or orthogonal – you are in; i.e., specify the input state for the circuit and say which case you should guess for each measurement outcome. (2) Which case can be decided correctly with certainty? (3) Assuming you were in this case (but, of course, did not know it), what is the probability of getting the measurement outcome that allows you to be certain? [3 marks]

(*b*) Let D denote the quantum Fourier transform (QFT) on $\mathbb{C}^M$.

(*i*) Write the matrix for the two-qubit QFT, in the computational basis. [3 marks]

(*ii*) Derive a simplified expression for D^2 in terms of outer products of the computational basis elements. [5 marks]

(*iii*) Suppose that the final two steps in a quantum algorithm on an M-dimensional quantum system are (1) to apply D^{-1} and (2) to measure the entire quantum system with respect to the computational basis. Suppose you perform all but the second-to-last step of the algorithm correctly, applying D instead of D^{-1} in the second-to-last step. Is your final measurement outcome still meaningful? Explain. [2 marks]