# 2009 Paper 6 Question 6

## Foundations of Functional Programming

(*a*) Define what it means for a $\lambda$-calculus term to be in normal form. Is it possible for a $\lambda$-term to have two normal forms that are not $\alpha$-equivalent? Provide justification for your answer. [3 marks]

(*b*) For each of the following, give an example of a $\lambda$-term that

(*i*) is in normal form;

(*ii*) is not in normal form but has a normal form; and

(*iii*) does not have a normal form.

For (*ii*), you should also present the term's normal form, and for (*iii*) you should show that the term does not have a normal form. [4 marks]

We define a $\lambda$-term $N$ to be *non-trivial* iff there exist $A$ and $B$ such that $N\,A \to^* \mathsf{true}$ and $N\,B \to^* \mathsf{false}$, where $\mathsf{true}$ and $\mathsf{false}$ encode the Booleans.

(*c*) Give an example of a $\lambda$-term that is non-trivial, and show that it is non-trivial. [2 marks]

We define a $\lambda$-term $N$ as *total* iff for each $\lambda$-term $M$, either $N\,M \to^* \mathsf{true}$ or $N\,M \to^* \mathsf{false}$

(*d*) Give an example of a $\lambda$-term that is total, and show that it is total. [2 marks]

(*e*) Prove that there is no non-trivial and total $\lambda$-term.

[Hint: Suppose $N$ is non-trivial and total where $N\,A \to^* \mathsf{true}$ and $N\,B \to^* \mathsf{false}$, and consider the term $N(Y\,L)$ where $L \equiv (\lambda x.\ \mathsf{if}\ (N\,x)\,B\,A)$ and where $Y$ is the fixed-point operator.]

[7 marks]

(*f*) What consequences does this have for defining a general equality $\lambda$-term such that

$$\begin{array}{llll} \mathsf{equal}\,A\,B & \to^* & \mathsf{true} & \text{if } A = B \\ \mathsf{equal}\,A\,B & \to^* & \mathsf{false} & \text{otherwise} \end{array}$$

[2 marks]