2007 Paper 8 Question 15

Denotational Semantics

(a) Suppose that $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$ are cpos.

(i) What properties does a function $f : D \to E$ need to satisfy in order to be continuous? [2 marks]

(ii) Assume also that $(C, \sqsubseteq_C)$ is a cpo and that $g : C \times D \to E$ is a continuous function. Let $g^* : C \to (D \to E)$ be defined by $g^*(c) = \lambda d \in D.\, g(c, d)$. Prove that g^* is continuous. You may refer to general facts about least upper bounds in product and function cpos provided that you state them clearly. [6 marks]

(b) Let $\mathbf{2} = (\{\bot, \top\}, \bot \sqsubseteq \top)$ be the unique domain with two elements.

(i) Draw a diagram which represents the elements of the function domain $\mathbf{2} \to \mathbf{2}$ and shows their ordering; [1 mark]

(ii) Any set X can be considered as a flat domain $X_\bot$ by adding a bottom element. Show that the strict continuous functions $X_\bot \to \mathbf{2}$ are in 1-1 correspondence with the subsets of X. [2 marks]

(c) Define what is meant by an admissible subset of a domain D. [2 marks]

(d) State the principle of Scott induction and prove its validity. [4 marks]

(e) Suppose that D is a domain and $f : D \times D \to D$ is a continuous function satisfying the property $\forall d, e \in D.\, f(d, e) = f(e, d)$. Let $g : D \times D \to D \times D$ be defined by $g(d_1, d_2) = (f(d_1, f(d_1, d_2)), f(f(d_1, d_2), d_2))$. Let $(u_1, u_2) = \mathit{fix}(g)$. Show that $u_1 = u_2$ using Scott induction. [3 marks]