2007 Paper 4 Question 9

Algorithms II

(a) Define *four* of the following terms, stating their defining properties and making use of equations where appropriate.

(i) flow network (a graph $G = (V, E)$);

(ii) flow (a function $f : V \times V \to \mathbb{R}$);

(iii) value of a flow (a real number);

(iv) residual network (a graph $G_f = (V, E_f)$);

(v) residual capacity (a function $c_f : V \times V \to \mathbb{R}$);

(vi) augmenting path (a sequence of edges).

[4 marks]

(b) Give some clear pseudocode for the Ford–Fulkerson method of finding the maximum flow and discuss its running time. Prove that, under appropriate conditions (which ones?), the method terminates. [4 marks]

(c) Given a flow network $G = (V, E)$ and two flows f_1 and f_2 in G, let $f_3 : V \times V \to \mathbb{R}$ be defined as

$$f_3(x, y) = f_1(x, y) + f_2(x, y).$$

Is f_3 a flow in G or not? Give a full proof of your answer, with reference to the three properties of a flow. [4 marks]

(d) Explain what a maximum matching in a bipartite graph is and explain how to solve it by transforming it into a maximum flow problem. [2 marks]

(e) Let $G(V, E)$ be a bipartite graph, with the vertex set V partitioned into a left subset L and a right subset R, and edges going from L to R. Let G' be the corresponding flow network according to the construction you explained in part (d). Derive a reasonably tight upper bound for the number of edges in any augmenting path that may be discovered by Ford–Fulkerson on G'. [6 marks]