2007 Paper 10 Question 6

Mathematics for Computation Theory

State the requirements for $(S, \leqslant)$ to be:

(a) a *partially ordered* set;

(b) a *totally ordered* set;

(c) a *well ordered* set. [5 marks]

Let $\mathbb{N}$ be the natural numbers. Give, without proof, three examples of relations $\leqslant_i$, where $i = 1, 2, 3$, such that $(\mathbb{N}, \leqslant_i)$ satisfies exactly i of the conditions (a), (b), (c). [3 marks]

Let $S = \{a, b\}$ be an alphabet, with total order $a < b$. Let $\Sigma = S^*$ be the set of all strings over S; for $w = s_1 s_2 \ldots s_n \in \Sigma$ we write $\ell(w) = n$, and for $1 \leqslant r \leqslant n = \ell(w)$ we write $w_r = s_1 s_2 \ldots s_r$. Denote by ε the unique word of Σ such that $\ell(\varepsilon) = 0$, the *null string*. Conventionally $w_0 = \varepsilon$ for all words $w \in \Sigma$.

Define relation $\prec$ on Σ as follows:

Let $v, w \in \Sigma$, and $n = \min\{\ell(v), \ell(w)\}$. Let $r = \max\{i \mid v_i = w_i\} \leqslant n$.

Then $v \prec w$ if:

either (i) $\ell(v) = r$;
or (ii) $v_{r+1} = v_r a, \quad w_{r+1} = w_r b, \quad$ where $v_r = w_r$.

Which of conditions (a), (b), (c) above are satisfied by $(\Sigma, \prec)$? [12 marks]