2006 Paper 12 Question 9

Numerical Analysis II

(*a*) In Peano's theorem, if a quadrature rule integrates polynomials of degree N exactly over an interval $[a, b]$, then the error in integrating $f \in C^{N+1}[a, b]$ is expressed as

$$E(f) = \int_a^b f^{(N+1)}(t)K(t)\, dt$$

where

$$K(t) = \frac{1}{N!} E_x[(x - t)_+^N].$$

Explain the notation $E(f)$, E_x, $(x - t)_+^N$. [4 marks]

(*b*) Assuming $x \in [a, b]$, and writing Taylor's theorem in the form

$$f(x) = P_N(x - a) + \frac{1}{N!} \int_a^x f^{(N+1)}(t)(x - t)^N\, dt$$

where P_N is a polynomial of degree N, prove Peano's theorem, explaining each step clearly. [8 marks]

(*c*) For the trapezium rule, what is N? [1 mark]

(*d*) If $K(t)$ does not change sign in $[a, b]$ then

$$E(f) = \frac{f^{(N+1)}(\xi)}{(N + 1)!} E(x^{N+1})$$

for some $\xi \in (a, b)$. Use this result to simplify

$$E(f) = \int_{-1}^1 f(x)\, dx - f(-1) - f(1).$$

[7 marks]