2005 Paper 10 Question 8

Mathematics for Computation Theory

State the requirements for $(S, \leqslant)$ to be:

(a) a *partially ordered* set;

(b) a *totally ordered* set;

(c) a *well ordered* set. [5 marks]

Let $(\mathbb{N}, \leqslant)$ be the natural numbers under the standard ordering. Define the *product ordering* $\leqslant_p$ on $(\mathbb{N} \times \mathbb{N})$ that is derived from this ordering. Which of conditions (a), (b), (c) does $\leqslant_p$ satisfy? [3 marks]

Let $(S, \leqslant)$ and $(T, \prec)$ be partially ordered sets, and $f : (S, \leqslant) \to (T, \prec)$ be a function. What condition must be satisfied in order that f be *monotonic*? [2 marks]

If f is a bijection, and both f and f^{-1} are monotonic, we say that $(S, \leqslant), (T, \prec)$ are *isomorphic* partially ordered sets.

Suppose that $(S, \leqslant)$ is a partially ordered set. A *topological sort* of $(S, \leqslant)$ is defined by specifying a total ordering $\sqsubseteq$ on S such that the identity map $\imath : (S, \leqslant) \to (S, \sqsubseteq)$ is monotonic.

Define *two* different topological sorts of $(\mathbb{N} \times \mathbb{N}, \leqslant_p)$, one of which is isomorphic to $\mathbb{N}$ with the standard ordering, while the other is not. Justify your claims. [10 marks]