2004 Paper 4 Question 7

Numerical Analysis I

(*a*) The mid-point rule can be expressed in the form

$$I_n = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(x)dx = f(n) + e_n$$

where

$$e_n = f''(\theta_n)/24$$

for some θ_n in the interval $(n-\frac{1}{2}, n+\frac{1}{2})$. Assuming that a formula for $\int f(x)dx$ is known, and using the notation

$$S_{p,q} = \sum_{n=p}^{q} f(n) \ ,$$

describe a method for estimating the sum of a slowly convergent series $S_{1,\infty}$, by summing only the first N terms and estimating the remainder by integration. [5 marks]

(*b*) Assuming that $f''(x)$ is a positive decreasing function, derive an estimate of the error $|E_N|$ in the method. [5 marks]

(*c*) Given

$$\int \frac{dx}{(1+x)\sqrt{x}} = 2\tan^{-1}\sqrt{x}$$

illustrate the method by applying it to

$$\sum_{n=1}^{\infty} \frac{1}{(1+n)\sqrt{n}} \ .$$

Verify that $f''(x)$ is positive decreasing for large x, and estimate the integral remainder to be added to $S_{1,N}$. [6 marks]

(*d*) How large should N be to achieve an absolute error of approximately 2×10^{-15}? [You may assume $N+1 \simeq N$ for this purpose.] [4 marks]