2003 Paper 13 Question 9

Numerical Analysis II

(a) Explain the term *positive semi-definite matrix.* [1 mark]

(b) Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices and let $\mathbf{x}$ be a vector of n elements. State *Schwarz's inequality* for each of the products $\mathbf{AB}$ and $\mathbf{Ax}$. What are the *singular values* of $\mathbf{A}$, and how are they related to the ℓ_2 norm of $\mathbf{A}$? [4 marks]

(c) Describe briefly the *singular value decomposition* $\mathbf{A} = \mathbf{UWV}^T$, and how it may be used to solve the linear equations $\mathbf{Ax} = \mathbf{b}$. [4 marks]

(d) Let $\hat{\mathbf{x}}$ be an approximate solution of $\mathbf{Ax} = \mathbf{b}$ and write $\mathbf{r} = \mathbf{b} - \mathbf{A}\hat{\mathbf{x}}$, $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$. Find an expression for an upper bound on the relative error $\|\mathbf{e}\| \,/\, \|\mathbf{x}\|$ in terms of computable quantities. Show how this formula may be computed using the singular values of $\mathbf{A}$. [8 marks]

(e) Suppose $\mathbf{A}$ is a 5×5 matrix and its singular values are 10^3, 1, 10^{-14}, 10^{-18}, 10^{-30}. If *machine epsilon* $\simeq 10^{-15}$ then choose a suitable *rank* for an approximate solution and form the generalised inverse $\mathbf{W}^+$. [3 marks]