COMPUTER SCIENCE TRIPOS Part IA

Monday 3 June 2002 1.30 to 4.30

Paper 1

Answer **two** *questions from Section A, and* **one** *question from* **each** *of Sections B, C, D and E.*

Submit the answers in six **separate** *bundles, each with its own cover sheet. On each cover sheet, write the numbers of* **all** *attempted questions, and circle the number of the question attached.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

1 Foundations of Computer Science

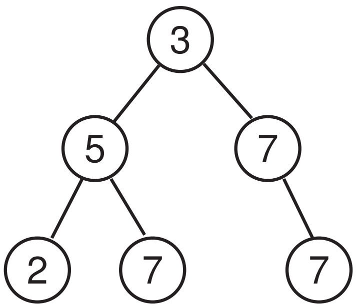
Consider the following ML declarations, involving binary trees:

```
datatype 'a tree = Lf
                 | Br of 'a * 'a tree * 'a tree;

exception E;

fun path Lf = raise E
  | path (Br(v,t1,t2)) = if v=7 then []
                         else 1 :: path t1
                              handle E => 2 :: path t2;
```

(*a*) The function `path` returns a path (a list of 1s and 2s) to an occurrence of the number 7 in the tree. Carefully explain how `path` works, taking the tree shown below as an example and indicating which occurrence of 7 will be found. [5 marks]



(*b*) Code the function `paths`, which returns the list of all paths to 7s in a binary tree. [5 marks]

2 Discrete Mathematics

(*a*) What is a *well-founded relation*? [2 marks]

(*b*) Let $\prec$ be a well-founded relation on a set A. Show that any non-empty subset S of A has a $\prec$-minimal element, i.e. an element $m \in S$ such that if $x \prec m$, then $x \notin S$, for all $x \in A$. [4 marks]

(*c*) Let a and b be distinct symbols. Using part (b), or otherwise, show that there is no string u such that $au = ub$. [4 marks]

3 Java

For each of the following language areas explain what Java provides and compare it with the nearest corresponding feature in ML (in each case viewing the languages in terms of the aspects of them covered in the Part IA lecture courses):

(*a*) the primitive data-types provided; [3 marks]

(*b*) exceptions: declaring, raising and handling them; [3 marks]

(*c*) having a function call, say `f(x)`, do different things based on the actual argument being one of several possible variations on some sort of data type. [4 marks]

4 Operating Systems

(*a*) What is an *interrupt*? [1 mark]

(*b*) A hardware device wishes to transfer information to the main memory of the computer for access by an application. The following three mechanisms are available:

(*i*) polled mode operation;

(*ii*) programmed I/O;

(*iii*) direct memory access (DMA).

For each one, summarise its operation and suggest an advantage it holds over the others.

[3 marks each]

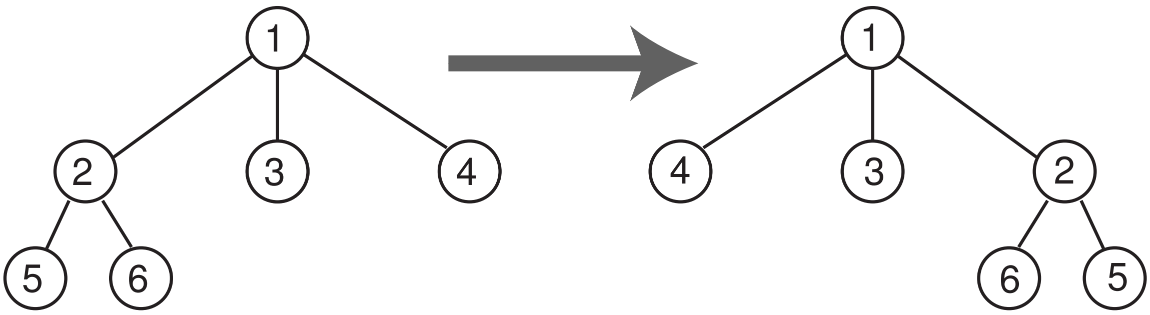
SECTION B

5 Foundations of Computer Science

This question concerns the following ML declaration of a tree datatype:

```
datatype 'a fan = Wave of 'a * ('a fan) list;
```

(*a*) Declare the function `flip`, which maps a tree to a mirror image of itself, as illustrated: [3 marks]



(*b*) Declare the curried function `paint f`, which copies a tree while applying the function `f` to each of its labels. [3 marks]

(*c*) Declare the function `same_shape`, which compares two trees and returns `true` if they are equal except for the values of their labels and otherwise returns `false`. [5 marks]

(*d*) State the types of functions `flip`, `paint` and `same_shape`. [3 marks]

(*e*) The function `paper` is declared in terms of the familiar functional `foldr`:

```
fun foldr f ([],    e) = e
  | foldr f (x::xs, e) = f(x, foldr f (xs,e));

fun paper (Wave(x,fs), q) = foldr paper (fs, q+1);
```

Describe the computation that results when `paper` is applied to a tree.

[6 marks]

6 Foundations of Computer Science

(a) Explain how O-notation is used to express efficiency of algorithms. [5 marks]

(b) Arrange the following list of complexity classes in order of decreasing efficiency in n. Briefly justify each relationship.

$$O(5n^2) \qquad O(e^n) \qquad O(n^{1/3}) \qquad O(n^3 - 3n^2) \qquad O(\log n) \qquad O(n2^n)$$

[4 marks]

(c) Suppose that f is a function from integers to integers such that $i \leqslant j$ implies $f(i) \leqslant f(j)$. Then there is an efficient algorithm to solve the equation $f(k) = y$, given the desired y and a range of values in which to search for k: the idea is repeatedly to halve this range. Code this algorithm as the ML function `search` whose arguments are f, y, and the range (a, b). Its result should be the greatest k such that $f(k) \leqslant y$ and $a \leqslant k \leqslant b$, provided such a k exists.
[11 marks]

SECTION C

7 Discrete Mathematics

State Fermat's Little Theorem and derive the Diffie–Hellman protocol for key exchange. [6 marks]

The protocol requires repeated multiplication (mod p), for some prime p, to achieve exponentiation. A procedure which avoids the potentially slow division by p after each multiplication to calculate the remainder is known as *Montgomery multiplication* ...

Given an odd prime p, let B be a power of 2 with $B > p$. Define $m(x) \equiv xB (\text{mod } p)$. Prove that:

- $m : \mathbb{Z}_p \to \mathbb{Z}_p$ is a bijection;
- $m(x \times y) = m^{-1}(m(x) \times m(y))$. [6 marks]

Given $u < pB$, let $v \equiv -up^{-1} (\text{mod } B)$ and $x = (u + vp) \div B$. If $x \geqslant p$, then subtract p from x. Prove that:

- x is an integer;
- $x \equiv uB^{-1} (\text{mod } p)$;
- $x < p$. [6 marks]

Deduce that $x = m^{-1}(u)$, observing that its calculation involves division only by B. [2 marks]

8 Discrete Mathematics

Let Ω be a set. Write $\mathcal{P}(\Omega)$ for its powerset. Recall the definition of the intersection of $\mathcal{B} \subseteq \mathcal{P}(\Omega)$:

$$\bigcap_{B \in \mathcal{B}} B = \{x \in \Omega \mid \forall B \in \mathcal{B}.\ x \in B\}\ .$$

(*a*) Let $\mathcal{B} \subseteq \mathcal{P}(\Omega)$ and $\mathcal{C} \subseteq \mathcal{P}(\Omega)$.

(*i*) Prove that

$$(\bigcap_{B \in \mathcal{B}} B) \cup (\bigcap_{C \in \mathcal{C}} C)\ \subseteq \bigcap_{(B,C) \in \mathcal{B} \times \mathcal{C}} (B \cup C)\ .$$

[3 marks]

(*ii*) Prove that

$$\bigcap_{(B,C) \in \mathcal{B} \times \mathcal{C}} (B \cup C)\ \subseteq\ (\bigcap_{B \in \mathcal{B}} B) \cup (\bigcap_{C \in \mathcal{C}} C)\ .$$

[6 marks]

(*b*) Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. Suppose that $\mathcal{A}$ is *intersection-closed* in the sense that

$$\text{if } \mathcal{B} \subseteq \mathcal{A}, \text{ then } \bigcap_{B \in \mathcal{B}} B \in \mathcal{A}\ .$$

Define

$$\mathcal{R} = \{(X, y) \in \mathcal{P}(\Omega) \times \Omega \mid \forall A \in \mathcal{A}.\ X \subseteq A \Rightarrow y \in A\}\ .$$

Let $C \subseteq \Omega$. Say C is $\mathcal{R}$-*closed* iff

$$\forall (X, y) \in \mathcal{R}.\ X \subseteq C \Rightarrow y \in C\ .$$

You are asked to show that the members of $\mathcal{A}$ are precisely the $\mathcal{R}$-closed subsets of Ω, in the following two stages:

(*i*) Show

$$\text{if } C \in \mathcal{A}, \text{ then } C \text{ is } \mathcal{R}\text{-closed}\ .$$

[2 marks]

(*ii*) Show

$$\text{if } C \text{ is } \mathcal{R}\text{-closed, then } C \in \mathcal{A}\ .$$

[Hint: Consider the set $\mathcal{B} = \{A \in \mathcal{A} \mid C \subseteq A\}$.] [9 marks]

SECTION D

9 Programming in Java

Observe that the matrix equation

$$\begin{pmatrix} f_{n+2} \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}$$

together with the initial conditions $f_0 = f_1 = 1$ defines the sequence of Fibonacci numbers.

Now observe that to compute f_n you need to multiply the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ by the matrix shown above raised to the power $n - 1$. In the special case where n is one bigger than a power of 2 this can be done by repeatedly squaring the matrix.

Write a Java program that will use this idea to compute f_{1025} given that $1024 = 2^{10}$. Your matrix multiplying or squaring code should be arranged to be potentially re-usable for matrices of arbitrary size: code that works only for 2 by 2 matrices is not acceptable.

Credit will be given for the clarity with which you present your design, the coherence of the explanation of how and why it works and for your comments about the cost or efficiency of your program.

[20 marks]

10 Programming in Java

Explain the following Java keywords. For each write a sentence giving an overview of what the word is about, then either write a second sentence that shows how it fits in with related concepts or write a small code-fragment to illustrate its purpose:

(*a*) `if`;

(*b*) `implements`;

(*c*) `import`;

(*d*) `instanceof`;

(*e*) `int`;

(*f*) `interface`;

(*g*) `package`;

(*h*) `private`;

(*i*) `public`;

(*j*) `protected`.

[2 marks each]

Please keep your explanation of each keyword brief so that the whole answer to this question remains a reasonable size: you are not expected to produce a detailed explanation or comprehensive illustrations.

SECTION E

11 Operating Systems

(*a*) Explain briefly the memory-management scheme of *paging*. [4 marks]

(*b*) Give *two* disadvantages of paging. [2 marks]

(*c*) A translation look-aside buffer (TLB) is sometimes used to optimise paging systems. Explain carefully how a TLB can be used in this way, and how it can optimise a paging system. [3 marks]

(*d*) The fictional Letni 2P chip uses (single-level) paging and has a memory access time of 8 nanoseconds and a TLB search time of 2 nanoseconds. What hit ratio (the probability that an item is in the TLB) must be achieved if we require an average (paged) memory access time of 12 nanoseconds? [4 marks]

(*e*) The management of the Letni Corporation wish you to design and evaluate a *multi-level* paging system for their new 64-bit processor, the 3P, which has 4K-sized pages.

(*i*) Give details of your proposed multi-level paging system. [5 marks]

(*ii*) State, and justify briefly, whether you think this proposal is realistic. [2 marks]

12 Operating Systems

Two important facilities provided by a conventional operating system (OS) are

- protection between different parts of the system,
- convenient programming interfaces by abstraction from the basic facilities exposed by the hardware.

Illustrate how these facilities are provided for the following four resources. In each case you should describe how protection is enforced and outline the interface that the hardware provides to the OS and an interface that the OS may provide to an application:

(*a*) CPU processing time; [5 marks]

(*b*) access to a device, such as a serial or parallel data port; [5 marks]

(*c*) storage of data and code in memory; [5 marks]

(*d*) storage of files on disk. [5 marks]

END OF PAPER