2001 Paper 1 Question 8

Discrete Mathematics

Let $(\mathrm{A}, \leqslant_{\mathrm{A}})$ and $(\mathrm{B}, \leqslant_{\mathrm{B}})$ be partially ordered sets.

(*a*) Define the *product order* on $\mathrm{A}\times\mathrm{B}$ and prove that it is a partial order. [4 marks]

The *upper bound* of a set $\mathrm{S} \subseteq \mathrm{A}$ is an element $u \in \mathrm{A}$ (but not necessarily in S) such that $\forall\, s \in \mathrm{S} \,.\, s \leqslant u$. The *least upper bound* of S is an upper bound of S that is less than every other upper bound of S. The *greatest lower bound* is defined similarly.

A *lattice* is a partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound.

(*b*) Prove that $(\mathbb{N}, |)$, the natural numbers under the divisibility order, form a lattice. [4 marks]

(*c*) Given a set X, prove that $(\mathcal{P}(\mathrm{X}), \subseteq)$, the power set of X under set inclusion, forms a lattice. [4 marks]

(*d*) Does every subset of $(\mathbb{N}, |)$ have a least upper bound and a greatest lower bound? Justify your answer. What about $(\mathbb{N}_0, |)$ and $(\mathcal{P}(\mathrm{X}), \subseteq)$? [4 marks]

(*e*) If $(\mathrm{A}, \leqslant_{\mathrm{A}})$ and $(\mathrm{B}, \leqslant_{\mathrm{B}})$ are lattices, show that $\mathrm{A}\times\mathrm{B}$ is a lattice under the product order. [4 marks]